FORCING TOTAL DETOUR MONOPHONIC SETS
IN A GRAPH

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Abstract. For a connected graph $G = (V, E)$ of order at least two, a total detour monophonic set of a graph $G$ is a detour monophonic set $S$ such that the subgraph induced by $S$ has no isolated vertices. The minimum cardinality of a total detour monophonic set of $G$ is the total detour monophonic number of $G$ and is denoted by $dm_t(G)$. A subset $T$ of a minimum total detour monophonic set $S$ of $G$ is a forcing total detour monophonic subset for $S$ if $S$ is the unique minimum total detour monophonic set containing $T$. A forcing total detour monophonic subset for $S$ of minimum cardinality is a minimum forcing total detour monophonic subset of $S$. The forcing total detour monophonic number $f_{tdm}(S)$ in $G$ is the cardinality of a minimum forcing total detour monophonic subset of $S$. The forcing total detour monophonic number of $G$ is $f_{tdm}(G) = \min \{f_{tdm}(S)\}$, where the minimum is taken over all minimum total detour monophonic sets $S$ in $G$. We determine bounds for it and find the forcing total detour monophonic number of certain classes of graphs. It is shown that for every pair $a, b$ of positive integers with $0 \leq a < b$ and $b > 2a+1$, there exists a connected graph $G$ such that $f_{tdm}(G) = a$ and $dm_t(G) = b$.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. For basic graph theoretic terminology we refer to Harary [2]. The distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u - v$ path in $G$. A $u - v$ path of length $d(u,v)$ is called a $u - v$ geodesic [1]. The
neighborhood of a vertex \( v \) is the set \( N(v) \) consisting of all vertices \( u \) which are adjacent with \( v \). The closed neighborhood of a vertex \( v \) is the set \( N[v] = N(v) \cup \{v\} \).

A vertex \( v \) is an extreme vertex if the subgraph induced by its neighbors is complete.

A chord of a path \( P \) is an edge joining two non-adjacent vertices of \( P \). A path \( P \) is called a monophonic path if it is a chordless path. A longest \( x - y \) monophonic path is called an \( x - y \) detour monophonic path. A set \( S \) of vertices of \( G \) is a detour monophonic set if each vertex \( v \) of \( G \) lies on an \( x - y \) detour monophonic path for some \( x, y \in S \). The minimum cardinality of a detour monophonic set of \( G \) is the detour monophonic number of \( G \) and is denoted by \( dm(G) \). The detour monophonic set of cardinality \( dm(G) \) is called \( dm \)-set. The detour monophonic number of a graph was introduced in [4] and further studied in [3].

A total detour monophonic set of a graph \( G \) is a detour monophonic set \( S \) such that the subgraph induced by \( S \) has no isolated vertices. The minimum cardinality of a total detour monophonic set of \( G \) is the total detour monophonic number of \( G \) and is denoted by \( dm_t(G) \). The total detour monophonic number of a graph was introduced and studied in [5].

For the graph \( G \) given in Figure 1.1, \( S_1 = \{u, v, x, y\} \), \( S_2 = \{u, v, y, z\} \), \( S_3 = \{t, u, x, y\} \) and \( S_4 = \{t, u, y, z\} \) are the minimum total detour monophonic sets of \( G \) and so \( dm_t(G) = 4 \).

A connected graph \( G \) may contain more than one minimum total detour monophonic sets. For example, the graph \( G \) given in Figure 1.1 contains four minimum total detour monophonic sets. For each minimum total detour monophonic set \( S \) in \( G \) there is always some subset \( T \) of \( S \) that uniquely determines \( S \) as the minimum total detour monophonic set containing \( T \). Such sets are called “forcing total detour monophonic subsets” and we discuss these sets in this paper.

The following theorems will be used in the sequel.

**Theorem 1.1.** [5] Each extreme vertex and each support vertex of a connected graph \( G \) belongs to every total detour monophonic set of \( G \). If the set \( S \) of all extreme vertices and support vertices form a total detour monophonic set, then it is the unique minimum total detour monophonic set of \( G \).

**Theorem 1.2.** [5] For the complete graph \( K_p(p \geq 2) \), \( dm_t(K_p) = p \).
Theorem 1.3. [5] For any non-trivial tree $T$, the set of all endvertices and support vertices of $T$ is the unique minimum total detour monophonic set of $G$.

Theorem 1.4. [5] For any connected graph $G$, $dm_t(G) = 2$ if and only if $G = K_2$.

Theorem 1.5. [4] No cutvertex of a connected graph $G$ belongs to any minimum detour monophonic set of $G$.

Throughout this paper $G$ denotes a connected graph with at least two vertices.

2. Forcing Total Detour Monophonic Sets

Definition 2.1. Let $G$ be a connected graph and let $S$ be a minimum total detour monophonic set of $G$. A subset $T$ of a minimum total detour monophonic set $S$ of $G$ is a forcing total detour monophonic subset for $S$ if $S$ is the unique minimum total detour monophonic set containing $T$. A forcing total detour monophonic subset for $S$ of minimum cardinality is a minimum forcing total detour monophonic subset of $S$. The forcing total detour monophonic number $f_{tdm}(S)$ in $G$ is the cardinality of a minimum forcing total detour monophonic subset of $S$. The forcing total detour monophonic number of $G$ is $f_{tdm}(G) = \min\{f_{tdm}(S)\}$, where the minimum is taken over all minimum total detour monophonic sets $S$ in $G$.

Example 2.1. For the graph $G$ given in Figure 1.1, $S_1 = \{u, v, x, y\}$, $S_2 = \{u, v, y, z\}$ and $S_3 = \{u, y, z\}$ are the minimum total detour monophonic sets of $G$. It is clear that $f_{tdm}(S_1) = 2$, $f_{tdm}(S_2) = 2$, $f_{tdm}(S_3) = 2$ and $f_{tdm}(S_1) = 2$ so that $f_{tdm}(G) = 2$. For the graph $G$ given in Figure 2.1, $S' = \{v_1, v_2, v_3\}$ and $S'' = \{v_1, v_3, v_4\}$ are the minimum total detour monophonic sets of $G$. Clearly, $f_{tdm}(S') = 1$ and so $f_{tdm}(G) = 1$.

![Figure 2.1: G](image)

The next theorem follows immediately from the definitions of the total detour monophonic number and forcing total detour monophonic number of a graph $G$.

Theorem 2.1. For a connected graph $G$, $0 \leq f_{tdm}(G) \leq dm_u(G) \leq p$.

Remark 2.1. The bounds in Theorem 2.1 are sharp. By Theorem 1.2, for the complete graph $K_p (p \geq 2)$, $dm_t(K_p) = p$, also $V(K_p)$ is the unique total detour monophonic set of $K_p$ and so $f_{tdm}(K_p) = 0$. The inequalities in Theorem 2.1 are strict. For the graph $G$ given in Figure 2.1, $dm_t(G) = 3$ and $f_{tdm}(G) = 1$. Thus $0 < f_{tdm}(G) < dm_t(G) < p$. 
The following theorem is an easy consequence of the definitions of the total detour monophonic number and forcing total detour monophonic number. In fact, the theorem characterizes graphs $G$ for which the lower bound in Theorem 2.1 is attained and also graphs $G$ for which $f_{tdm}(G) = 0$ and $f_{tdm}(G) = dm_t(G)$.

**Theorem 2.2.** Let $G$ be a connected graph. Then

(i) $f_{tdm}(G) = 0$ if and only if $G$ has a unique minimum total detour monophonic set.

(ii) $f_{tdm}(G) = 1$ if and only if $G$ has at least two minimum total detour monophonic sets, one of which is a unique minimum total detour monophonic set containing one of its elements, and

(iii) $f_{tdm}(G) = dm_t(G)$ if and only if no minimum total detour monophonic set of $G$ is the unique minimum total detour monophonic set containing any of its proper subsets.

**Definition 2.2.** A vertex $v$ of a connected graph $G$ is said to be a total detour monophonic vertex of $G$ if $v$ belongs to every minimum total detour monophonic set of $G$.

We observe that if $G$ has a unique minimum total detour monophonic set $S$, then every vertex in $S$ is a total detour monophonic vertex of $G$. Also, if $x$ is an extreme vertex of $G$ or a support vertex of $G$, then $x$ is a total detour monophonic vertex of $G$. For the graph $G$ given in Figure 2.1, $v_1$ and $v_3$ are the total detour monophonic vertices of $G$.

The following theorem and corollary follows immediately from the definitions of total detour monophonic vertex and forcing total detour monophonic subset of $G$.

**Theorem 2.3.** Let $G$ be a connected graph and let $\mathcal{U}_{dm}$ be the set of relative complements of the minimum forcing total detour monophonic subsets in their respective minimum total detour monophonic sets in $G$. Then $\bigcap_{F \in \mathcal{U}_{dm}} F$ is the set of total detour monophonic vertices of $G$.

**Corollary 2.1.** Let $G$ be a connected graph and let $S$ be a minimum total detour monophonic set of $G$. Then no total detour monophonic vertex of $G$ belongs to any minimum forcing total detour monophonic subset of $S$.

**Theorem 2.4.** Let $G$ be a connected graph and let $M$ be the set of all total detour monophonic vertices of $G$. Then $f_{tdm}(G) \leq dm_t(G) - |M|$.

**Proof.** Let $S$ be any minimum total detour monophonic set of $G$. Then $dm_t(G) = |S|$, $M \subseteq S$ and $S$ is the unique minimum total detour monophonic set containing $S - M$. Thus $f_{tdm}(G) \leq |S - M| = |S| - |M| = dm_t(G) - |M|$. □

**Corollary 2.2.** If $G$ is a connected graph with $l$ extreme vertices and $k$ support vertices, then $f_{tdm}(G) \leq dm_t(G) - (l + k)$. 
Remark 2.2. The bound in Theorem 2.4 is sharp. For the graph $G$ given in Figure 2.1, $d_{tdm}(G) = 3$ and $f_{tdm}(G) = 1$. Also, $M = \{v_1, v_3\}$ is the set of all total detour monophonic vertices of $G$ and so $f_{tdm}(G) = d_{tdm}(G) - |M|$. Also the inequality in Theorem 2.4 can be strict. For the graph $G$ given in Figure 2.2, $S_1 = \{u, v, w\}$ and $S_2 = \{x, y, w\}$ are the minimum total detour monophonic sets of $G$ and so $d_{tdm}(G) = 3$. Since $S_1$ is the unique minimum total detour monophonic set contains the subset $\{u\}$ so that $f_{tdm}(S_1) = 1$ and $S_2$ is the unique minimum total detour monophonic set contains the subset $\{x\}$ so that $f_{tdm}(S_2) = 1$. Hence, we have $f_{tdm}(G) = 1$. Also, the vertex $w$ is the unique total detour monophonic vertex of $G$, we have $f_{tdm}(G) < d_{tdm}(G) - |M|$. 

Theorem 2.5. Let $G$ be a connected graph and let $S$ be a minimum total detour monophonic set of $G$. Then no cutvertex of $G$ (which is not a support vertex) belongs to any minimum forcing total detour monophonic subset of $S$.

Proof. Let $v$ be a cutvertex of $G$ which is not a support vertex. By Theorems 1.1 and 1.5, $v$ does not belong to any minimum total detour monophonic set of $G$. Since any minimum forcing total detour monophonic subset of $S$ is a subset of $S$, then it is clear. □

The next result follows from Theorem 1.4.

Theorem 2.6. If $G$ is a connected graph with $d_{tdm}(G) = 2$, then $f_{tdm}(G) = 0$.

Now, we proceed to determine the forcing total detour monophonic number of certain classes of graphs.

Theorem 2.7. For any cycle $C_n (n \geq 4)$, $f_{tdm}(C_n) = 3$.

Proof. Let $C_n : v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_n, v_1$ be a cycle of order $n$. It is easily observed that any three consecutive vertices of $C_n$ is a minimum total detour monophonic set of $C_n$. Then clearly no minimum total detour monophonic set of $C_n$ is the unique minimum total detour monophonic set containing any of its proper subsets. Hence by Theorem 2.2(iii), $f_{tdm}(C_n) = 3$. □

Theorem 2.8. For any complete graph $G = K_p (p \geq 2)$ or any non-trivial tree $G = T$, $f_{tdm}(G) = 0$. 
Proof. For \( G = K_p \), it follows from Theorem 1.2 that the set of all vertices of \( G \) is the unique minimum total detour monophonic set of \( G \). Now, it follows from Theorem 2.2 (i) that \( f_{tdm}(G) = 0 \). If \( G \) is a non-trivial tree, then by Theorem 1.3, the set of all endvertices and support vertices of \( G \) is the unique minimum total detour monophonic set of \( G \) and so by Theorem 2.2 (i), \( f_{tdm}(G) = 0 \). □

**Theorem 2.9.** For the complete bipartite graph \( G = K_{m,n}(m, n \geq 2) \),

\[
f_{tdm}(G) = \begin{cases} 1 & \text{if } 2 = m < n \\ 3 & \text{if } 2 = m = n \\ 4 & \text{if } 3 \leq m \leq n \\ \end{cases}
\]

**Proof.** Let \( U = \{u_1, u_2, \ldots, u_m\} \) and \( W = \{w_1, w_2, \ldots, w_n\} \) be the bipartition of \( G \), where \( m \leq n \). We prove this theorem by considering four cases.

**Case 1.** \( 2 = m = n \). Since \( G \) is a cycle of order 4, by Theorem 2.7, we have \( f_{tdm}(G) = 3 \).

**Case 2.** \( 2 = m < n \). For any \( j(1 \leq j \leq n) \), \( S_j = U \cup \{w_j\} \) is a minimum total detour monophonic set of \( G \). Since \( n \geq 3 \), then by Theorem 2.2(ii), we have \( f_{tdm}(G) = 1 \).

**Case 3.** If \( 3 = m = n \), then any minimum total detour monophonic set of \( G \) is of the following forms: (i) \( U \cup \{w_j\} \) for some \( j(1 \leq j \leq n) \), (ii) \( W \cup \{u_i\} \) for some \( i(1 \leq i \leq m) \), or (iii) any set got by choosing any two elements from each of \( U \) and \( W \). If \( 3 = m < n \), then any minimum total detour monophonic set of \( G \) is either \( U \cup \{w_j\} \) for some \( j(1 \leq j \leq n) \), or any set got by choosing any two elements from each of \( U \) and \( W \). Hence in both cases, we have \( d_{tdm}(G) = 4 \). Clearly, no minimum total detour monophonic set of \( G \) is the unique minimum total detour monophonic set containing any of its proper subsets. Then by Theorem 2.2(iii), we have \( f_{tdm}(G) = d_{tdm}(G) = 4 \).

**Case 4.** \( 4 = m \leq n \). Then any minimum total detour monophonic set is got by choosing any two elements from each of \( U \) and \( W \), and \( G \) has at least two minimum total detour monophonic sets. Hence \( d_{tdm}(G) = 4 \). Clearly, no minimum total detour monophonic set of \( G \) is the unique minimum total detour monophonic set containing any of its proper subsets. Then by Theorem 2.2(iii), we have \( f_{tdm}(G) = d_{tdm}(G) = 4 \). □

**Theorem 2.10.** For every pair \( a, b \) of positive integers with \( 0 \leq a < b \) and \( b > 2a + 1 \), there exists a connected graph \( G \) such that \( f_{tdm}(G) = a \) and \( d_{tdm}(G) = b \).

**Proof.** If \( a = 0 \), let \( G = K_b \). Then by Theorem 2.8, \( f_{tdm}(G) = 0 \) and by Theorem 1.2, \( d_{tdm}(G) = b \). Thus we assume that \( 0 < a < b \).

For each \( i \) with \( 1 \leq i \leq a \), let \( C_i : u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4}, u_{i,1} \) be a cycle of order 4. Let \( K_{1,b-2a-1} \) be a star with the cutvertex \( x \) and \( V(K_{1,b-2a-1}) = \{x, v_1, v_2, \ldots, v_{b-2a-1}\} \). The graph \( G \) is obtained from \( C_i(1 \leq i \leq a) \) and \( K_{1,b-2a-1} \) by joining the vertices \( x \) and \( u_{i,1} \); and by joining the vertices \( u_{i,2} \) and \( u_{i,4} \) for \( 1 \leq i \leq a \). The graph \( G \) is shown in Figure 2.3. Let \( S = \{v_1, v_2, \ldots, v_{b-2a-1}, u_{1,3}, u_{2,3}, \ldots, u_{a,3}, x\} \) be the set of all extreme vertices and support vertex of \( G \). By Theorem 1.1, every total detour monophonic set of \( G \) contains \( S \). It is easily
verified that $S$ is not a total detour monophonic set of $G$. We observe that every minimum total detour monophonic set of $G$ contains exactly one vertex from $\{u_{i,2}, u_{i,4}\}$ for every $i(1 \leq i \leq a)$. Hence $dm_t(G) \geq b - a + a = b$. On the other hand, $S' = S \cup \{u_{1,2}, u_{2,2}, \cdots, u_{a,2}\}$ is a total detour monophonic set of $G$, it follows that $dm_t(G) \leq b$. Thus $dm_t(G) = b$.

Next we show that $f_{tdm}(G) = a$. It is observed that $S$ is the set of all total detour monophonic vertices of $G$. Then by Theorem 2.4, $f_{tdm}(G) \leq dm_t(G) - |S| = b - (b - a) = a$. Now, since $dm_t(G) = b$ and every minimum total detour monophonic set of $G$ contains $S$, it is easily seen that every minimum total detour monophonic set $S_1$ of $G$ is of the form $S \cup \{x_1, x_2, \cdots, x_a\}$, where $x_i \in \{u_{i,2}, u_{i,4}\}$ for every $i(1 \leq i \leq a)$. Let $T$ be any proper subset of $S_1$ with $|T| < a$. Then there is a vertex $x \in S_1 - S$ such that $x \notin T$. If $x = u_{i,2}(1 \leq i \leq a)$, then $S_2 = (S_1 - \{u_{i,2}\}) \cup \{u_{i,4}\}$ is a minimum total detour monophonic set of $G$ containing $T$. Similarly, if $x = u_{i,4}(1 \leq i \leq a)$, then $S_3 = (S_1 - \{u_{i,4}\}) \cup \{u_{i,2}\}$ is a minimum total detour monophonic set of $G$ containing $T$. Thus $S_1$ is not the unique minimum total detour monophonic set of $G$ containing $T$ and so $T$ is not a forcing total detour monophonic subset of $S_1$. Since this is true for all minimum total detour monophonic sets of $G$, it follows that $f_{tdm}(G) \geq a$ and so $f_{tdm}(G) = a$. $\square$
References


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