LOCAL CONNECTIVE CHROMATIC NUMBER OF DIRECT PRODUCT OF PATHS AND CYCLES

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Abstract. Graph coloring is one of the most important concepts in graph theory. There are many types of coloring. We study on the local connective chromatic number of a graph $G$ that is defined by us. In this paper, we determine the local connective chromatic number of the direct product of two paths $P_m \times P_n$, two cycles $C_m \times C_n$ and for the direct product of a cycle and a path $C_m \times P_n$, where $m$ and $n$ are the number of vertices.

1. Introduction

Let $G$ be a simple undirected graph, where $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of $G$, respectively. For two vertices $u, v \in V(G)$, $u$ and $v$ are adjacent if they are joined by an edge. Two vertices that are not adjacent in a graph $G$ are said to be independent. The independence number $\beta(G)$ of a graph $G$ is the maximum cardinality among the independent sets of vertices of $G$. For the notations and terminology not defined here, we follow [6].

The connectivity $\kappa = \kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected or trivial graph. Two paths are internally disjoint (vertex disjoint) if they do not share a common vertex except their end vertices. The local connectivity $\kappa_G(u, v) = \kappa(u, v)$ between two distinct vertices $u$ and $v$ of a graph $G$ is defined as the smallest number of vertices whose removal separates $u$ and $v$. By Menger’s theorem [14], $\kappa(u, v)$ equals the maximum number of internally disjoint $u - v$ paths in $G$ and $\kappa(G) = \min\{\kappa(u, v) : u, v \in V(G)\}$. It is straightforward to verify that $\kappa(G) \leq \delta(G)$ and $\kappa(u, v) \leq \min\{\deg(u), \deg(v)\}$ [16].

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The local connective coloring is defined by us by inspiring the notion of packing coloring [5, 9, 12, 18].

Routing is the process of delivering messages among vertices and selecting the best paths in a network. Efficiency and reliability of routing can be achieved by using internally disjoint paths because the failure of a path would not affect the performance of other paths. Then the more internally disjoint paths are the better for a network [13]. Thus, we use the term internally disjoint path in our coloring and coloring the vertices depending on the number of internally disjoint paths between two vertices.

A graph $G$ which has a local connective $k$-coloring can be partitioned into disjoint color classes $X_1, X_2, ..., X_k$ and can be drawn as a $k$-partite graph. Thereby, the graph is partitioned into the subsets which have disjoint paths. Looking for a secure disjoint path between two vertices $u$ and $v$ in any color class $X_i$, we make this search with the vertices in the other color classes. This indicate that we look for disjoint paths starting from $u$ and ending to $v$ using the vertices in the other color classes. Thus, this search can be made with $V(G) - (|X_i| - 2)$ vertices. Thereby, NP-complete problem can be solved more easily. Local connective coloring provides to facilitate the routing of non-adjacent vertices to communicate with each other.

The direct product $G \times H$ of two graphs $G$ and $H$ is a graph with $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E(G) \text{ and } v_1v_2 \in E(H)\}$. It is also known as Kronecker product, tensor product, categorical product and graph conjunction. This graph product is commutative and associative [3]. The direct product of graphs has been extensively investigated concerning graph recognition and decomposition, graph embeddings, matching theory and stability in graphs [1, 4]. More generally, the direct product is a widely used tool in the area of graph colorings [11].

**Lemma 1.1.** [17] Let $G$ be a connected graph. If $G$ has no odd cycle, then $G \times K_2$ has exactly two connected components isomorphic to $G$.

**Theorem 1.1.** [17] Let $G$ and $H$ be connected graphs. The graph $G \times H$ is connected if and only if any $G$ or $H$ contains an odd cycle.

**Corollary 1.1.** [17] If $G$ and $H$ are connected graphs with no odd cycles then $G \times H$ has exactly two connected components.

**Theorem 1.2.** [15] Let $G = (V, E)$ be a connected graph, and $H = (V_1, V_2, E')$ be a bipartite connected graph, then $G \times H$ is a bipartite graph, the partition of the vertex set is $(V \times V_1)$ and $(V \times V_2)$.

**Theorem 1.3.** [8] The direct product of two connected graphs is a non-connected graph if and only if both are bipartite.

**Lemma 1.2.** [10] If $G = (V_0 \cup V_1, E)$ and $H = (W_0 \cup W_1, F)$ are bipartite graphs, then $(V_0 \times W_0) \cup (V_1 \times W_1)$ and $(V_0 \times W_1) \cup (V_1 \times W_0)$ are vertex sets of the two components of $G \times H$.

**Lemma 1.3.** [10] If $G$ is a connected, bipartite graph and $n \geq 4$ is an even integer, then the graph $G \times C_n$ consists of two isomorphic connected components.
Theorem 1.4. [2] If $G$ and $H$ are regular graphs then $G \times H$ is also a regular graph.

2. Local Connective Chromatic Number of Direct Product Graphs

Definition 2.1. A local connective $k$-coloring of a graph $G$ is a mapping $c : V(G) \rightarrow \{1, 2, ..., k\}$ such that

1. If $uv \in E(G)$, then $c(u) \neq c(v)$, and
2. If $uv \notin E(G)$ and $\kappa(u,v) \geq i$, then $c(u) = c(v) = i$, where $\kappa(u,v)$ is the maximum number of internally disjoint paths between $u$ and $v$.

The smallest integer $k$ for which there exists a local connective $k$-coloring of $G$ is called the local connective chromatic number of $G$, and it is denoted by $\chi_{lc}(G)$.

The first condition characterizes proper coloring. Thus, every local connective coloring is a proper coloring.

The vertices of $G$ are partitioned into disjoint color classes $X_1, X_2, ..., X_k$, where each color class $X_i$ consists of distinct vertices $u, v \in X_i$ such that $\kappa(u,v) \geq i$ and $\bigcup_{i=1}^{k} X_i = V(G)$. The maximum cardinality of $X_i$ in $G$ is denoted by $k_i$.

In this section, we give local connective chromatic number of direct product of paths and cycles. Let $G$ and $H$ be any two graphs with vertex sets $V(G) = \{v_1, v_2, ..., v_m\}$, $V(H) = \{v_1, v_2, ..., v_n\}$, respectively. A vertex $(u_i, v_j)$ is abbreviated as $w_{ij}$, where $w_{ij} \in V(G \times H)$, $i \in \{1, 2, ..., m\}$, $j \in \{1, 2, ..., n\}$.

Theorem 2.1. Let $P_m$ and $P_n$ be two paths of order $m$ and $n$, respectively. Then, $4 \leq \chi_{lc}(P_m \times P_n) \leq \max\{2n + 5, m + n + 7\}$ for $m \leq n$.

Proof. It is known that paths are bipartite graphs. By Theorem 1.3, the graph $P_m \times P_n$ is non-connected for $m$ and $n$ being odd or even. Further by Corollary 1.1, $P_m \times P_n$ has two connected components as $G_1$ and $G_2$. Since $\delta(G_1) = \delta(G_2) = 2$ and $\Delta(G_1) = \Delta(G_2) = 4$, we have $2 \leq \kappa(w_{ij}, w_{kl}) \leq 4$, where $w_{ij}, w_{kl} \in V(G_1)$ or $V(G_2)$, $i, k \in \{1, 2, ..., m\}$, $j, l \in \{1, 2, ..., n\}$. Thus, $k_i \leq 1$ for $i \geq 5$. That is, the pair of vertices can be colored with the same color at most color 4, and the remaining uncolored vertices receive different colors.

We prove this theorem in four cases for $m$ and $n$ being odd or even.

Case 1. Let $m$ and $n$ be odd.

Since $|V(P_m \times P_n)| = mn$, we have $|V(G_1)| = \lceil \frac{mn}{2} \rceil$, $|V(G_2)| = \lfloor \frac{mn}{2} \rfloor$. The graph $P_m \times P_n$ has four vertices of each of degree one in the only one component. Assume that these vertices be in the component $G_1$. Since there is one internally disjoint path between these vertices, they can be colored with color 1. The vertex $w_{ij}$ can be colored with color 1, where $i \in \{1, 3, ..., m\}$, $j \in \{1, 3, ..., n\}$. Thus, $\beta(G_1) = \lceil \frac{mn}{2} \rceil$ vertices in $G_1$ are colored with color 1, and $\lfloor \frac{mn}{2} \rfloor$ vertices in $G_1$ remain uncolored. $\lfloor \frac{mn}{2} \rfloor < \lceil \frac{mn}{2} \rceil$ and thus color the graph $G_2$ with color 2. When we start coloring from the vertex $w_{12}$ and color all vertices which are not adjacent with each other in $G_2$, then maximum $\lfloor \frac{mn}{2} \rfloor$ vertices in $G_2$ are colored.
with color 2. Thus, there are $\lceil \frac{n}{2} \rceil \lfloor \frac{m}{2} \rfloor$ and $n \lfloor \frac{m}{2} \rfloor$ uncolored vertices in $G_2$ and $G$, respectively.

Case 1.1. Let $m = 3$ and $n \geq 3$.

For all vertices $w_{ij}, w_{kl}$ in $G_1$ or $G_2$, we have $\kappa(w_{ij}, w_{kl}) \leq 2$, where $i, k \in \{1, 2, 3\}$, $j, l \in \{1, 2, \ldots, n\}$. Thus, there is not any vertex colored with color 3 and color 4. Hence, the remaining $n \lfloor \frac{m}{2} \rfloor$ vertices are colored with different colors. Then we have $\chi_{lc}(P_3 \times P_n) = 2 + n \lfloor \frac{m}{2} \rfloor = n + 2$.

Case 1.2. Let $m = 5$ and $n \geq 5$.

Since $\kappa(G_1) = 1$, $\kappa(G_2) = 2$, we have $\kappa(w_{ij}, w_{kl}) \geq 2$, where $w_{ij}, w_{kl} \in V(G_2)$. Four vertices in $P_5 \times P_n$ are colored with color 3. Color the vertex $w_{ij}$ in $G_2$ with color 3 for the minimum local connective coloring number, where $i \in \{2, 4\}$, $j \in \{3, 5\}$.

Case 1.2.1. If $m = n = 5$, no vertices that remain uncolored in $G_1$ or $G_2$ are colored with color 4. Thus, the remaining vertices receive different colors and we have $\chi_{lc}(P_5 \times P_5) = 3 + n \lfloor \frac{m}{2} \rfloor - 4 = 9$.

Case 1.2.2. If $m = 5$ and $n \geq 7$, the vertex $w_{ij}$ for $i \in \{2, 4\}$ in $G_1$ is colored with color 4. The remaining $n \lfloor \frac{m}{2} \rfloor - 6 = 2n - 6$ vertices receive different colors. Thus, we have $\chi_{lc}(P_5 \times P_n) = 2n - 2$.

Case 1.3. Let $m \geq 7$ and $n \geq 7$.

Take the vertex $w_{ij}$ in $G_1$, where $i \in \{2, 4, \ldots, m - 1\}, j \in \{2, 4, \ldots, n - 1\}$. The number of internally disjoint paths between these vertices is at most 4. Thus, maximum $(\frac{m-1}{2})(\frac{n-1}{2})$ vertices are colored with color 3, and there is no vertex in $G_1$ remains uncolored.
Case 1.3.1. If \( m = n = 7 \), the number of internally disjoint paths between only two vertices in \( G_2 \) is 4. Hence, these vertices receive color 4, and we have \( \chi_{lc}(P_7 \times P_7) = 4 + \lfloor \frac{m}{2} \rfloor \lceil \frac{n}{2} \rceil - 2 = 14 \).

Case 1.3.2. Let \( m = 7 \) and \( n \geq 9 \). Since \( m = 7 \), there are four vertices in \( G_2 \) that the number of internally disjoint paths between them is 4. Thus, these four vertices receive color 4, and we have \( \chi_{lc}(P_7 \times P_n) = 4 + \lfloor \frac{m}{2} \rfloor \lceil \frac{n}{2} \rceil - 4 = \frac{3}{2} + (n + 1) \).

Case 1.3.3. Let \( m \geq 9 \) and \( n \geq 9 \).

Take the vertices \( w_{ij} \), where \( i \in \{4, 6, 8, \ldots, m - 3\}, j \in \{3, 5, 7, \ldots, n - 2\} \) and \( w_{kl} \), where \( k \in \{2, m - 1\}, l \in \{5, 7, 9, \ldots, n - 4\} \) in \( G_2 \). Since the number of internally disjoint paths between them is 4, \( \lfloor \frac{m-4}{2} \rfloor \lceil \frac{n-3}{2} \rceil + 2 \lfloor \frac{n-6}{2} \rfloor \) vertices are colored with color 4. Hence, the remaining \( m + 3 \) vertices receive \( m + 3 \) different colors, and we have \( \chi_{lc}(P_m \times P_n) = 4 + m + 3 = m + 7 \).

Consequently, if \( m \) and \( n \) are odd, then we have

\[
\chi_{lc}(P_m \times P_n) = \begin{cases} 
  n + 2, & m = 3, n \geq 3 \\
  9, & m = n = 5 \\
  2n - 2, & m = 5, n \geq 7 \\
  14, & m = n = 7 \\
  \frac{1}{2}(n + 1), & m = 7, n \geq 9 \\
  m + 7, & m \geq 9, n \geq 9.
\end{cases}
\]

Case 2. Let \( m \) be even and \( n \) be odd.

\( |V(P_m \times P_n)| = mn \) and \( |V(G_1)| = |V(G_2)| = \frac{mn}{2} \). The graph \( P_m \times P_n \) has four vertices of each of degree one. Two of them are in \( G_1 \) and the other two vertices are in \( G_2 \). Assume that we start coloring from the vertex \( w_{21} \) in \( G_1 \). Assign color 1 to every vertex when \( i \) and \( j \) are odd. Thus, \( \beta(G_1) = \frac{m}{2} \lfloor \frac{n}{2} \rfloor \) vertices are colored with color 1. The graph \( G_1 \) has \( \lfloor \frac{m}{2} \rfloor \) uncolored vertices of degree 2 and \( \lfloor \frac{n}{2} \rfloor (\frac{n}{2} - 1) \) uncolored vertices each of degree 4. Then the total number of the remaining uncolored vertices in \( G_1 \) is \( \lfloor \frac{n}{2} \rfloor \frac{m}{2} \).

Start coloring the graph \( G_2 \) with color 2 for the minimum number of local connective coloring. Assume that we start coloring from the vertex \( w_{21} \). The graph \( G_2 \) has total \( \frac{mn}{2} - 2 \) vertices of degree two and four, and since the number of internally disjoint paths between these vertices is at least 2, each vertex which is not adjacent with each other is colored with color 2. Thus, the vertices \( w_{ij} \), where \( i \in \{2, 4, 6, \ldots, m - 2\}, j \in \{1, 3, 5, \ldots, n\} \) and \( w_{kl} \), where \( k \in \{3, 5, 7, \ldots, n - 2\} \) are colored with color 2. Then maximum \( \lfloor \frac{m}{2} \rfloor (\frac{m}{2} - 2) + \frac{n-3}{2} = \frac{m}{2} \lfloor \frac{n}{2} \rfloor - 2 \) vertices in \( G_2 \) receive color 2. Among the remaining \( 2 + \lfloor \frac{m}{2} \rfloor \frac{m-2}{2} \) vertices in \( G_2 \), two of them have degree one, \( \lfloor \frac{m}{2} \rfloor \) of them are each of degree 2 and \( \lfloor \frac{m}{2} \rfloor (\frac{n}{2} - 1) \) of them are each of degree 4. Thus, there are \( 2 + \lfloor \frac{m}{2} \rfloor m \) vertices in the graph \( P_m \times P_n \) remain uncolored.

Case 2.1. Let \( m = 2 \) and \( n \geq 3 \).

In this case, \( P_n \cong G_1, P_n \cong G_2 \) and \( P_2 \times P_n \cong 2P_n \). Since \( \chi_{lc}(P_n) = 1 + \lfloor \frac{n}{2} \rfloor \).
by [7], we get

$$\chi_{lc}(P_2 \times P_n) = n + \chi_{lc}(P_n) = n + 1 + \left\lfloor \frac{n}{2} \right\rfloor = \frac{3n + 1}{2}.$$ 

**Case 2.2.** Let $m = 4$ and $n \geq 5$.

Since $\kappa(w_{ij}, w_{kl}) \leq 2$, where $w_{ij}, w_{kl} \in V(G_1)$ (or $V(G_2)$) there is not any vertex colored with color 3 and 4. Hence, the remaining $m\left\lfloor \frac{n}{2} \right\rfloor + 2 = 2n$ vertices receive different colors. Then we have $\chi_{lc}(P_4 \times P_n) = 2n + 2$.

**Case 2.3.** Let $m \geq 6$ and $n \geq 7$.

Every pair of vertices in $G_1$ each of degree 4 satisfy the condition $\kappa(w_{ij}, w_{kl}) \geq 3$, where $i, k \in \{2, 4, \ldots, m - 2\}$, $j, l \in \{2, 4, \ldots, n - 1\}$. Thus, $\left\lfloor \frac{n}{2} \right\rfloor \left(\frac{n}{2} - 1\right)$ vertices are colored with color 3. The number of the remaining vertices in $G_1$ is $\left\lceil \frac{n}{2} \right\rceil$ and each of them has degree two. Further, these vertices satisfy the condition $\kappa(w_{ij}, w_{kl}) \leq 2$, where $j, l \in \{2, 4, \ldots, n - 1\}, j \neq l$. Then they receive different colors. Hence, we have $\chi_{lc}(G_1) = 2 + \left\lfloor \frac{n}{2} \right\rfloor$.

Consider coloring the graph $G_2$. For the vertices $w_{ij}$, where $i \in \{5, 7, 9, \ldots, m - 3\}, j \in \{2, 4, 6, \ldots, n - 1\}$ and $w_{kl}$, where $k \in \{3, m - 1\}, l \in \{4, 6, 8, \ldots, n - 3\}$, the number of internally disjoint paths between them are 4 and so $\left\lceil \frac{m - 3}{2} \right\rceil \left(\frac{n - 1}{2} - 2\right)$ vertices are colored with color 4. The remaining $\frac{n+11}{2}$ vertices in $G_2$ receive different colors. Thus, $\chi_{lc}(G_2) = 2 + \frac{n+11}{2}$, and we have

$$\chi_{lc}(P_m \times P_n) = \chi_{lc}(G_1) + \chi_{lc}(G_2) = n + 9.$$

Consequently, if $m$ is even and $n$ is odd, then we have

$$\chi_{lc}(P_m \times P_n) = \begin{cases} \frac{3n+1}{2}, & m = 2, n \geq 3 \\ 2n + 2, & m = 4, n \geq 5 \\ n + 9, & m \geq 6, n \geq 7. \end{cases}$$

**Case 3.** Let $m$ be odd and $n$ be even.

The graph $P_m \times P_n$ has four vertices of each of degree one. Two of them are in $G_1$ and the other two vertices are in $G_2$. If we color the vertices of $G_1$ starting from the vertex $w_{11}$ as Case 2, $\left\lceil \frac{m}{2} \right\rceil$ vertices receive color 1. For color 2, assume that we start coloring from the vertex $w_{12}$ in $G_2$. The vertices $w_{ij}$, where $i \in \{5, 7, \ldots, m - 2\}, j \in \{2, 4, 6, \ldots, n\}$ and $w_{kl}$, where $k \in \{1, m\}, l \in \{2, 4, 6, \ldots, n - 2\}$ are colored with color 2. There are at most $\left\lceil \frac{(m-3)n}{4} \right\rceil = \left\lceil \frac{m}{2} \right\rceil \frac{n}{2} - 2$ vertices which receive color 2. Thus, $\left\lceil \frac{m}{2} \right\rceil n + 2$ vertices remain uncolored in $P_m \times P_n$.

**Case 3.1.** Let $m = 3$ and $n \geq 4$.

In this case, $\kappa(w_{ij}, w_{kl}) \leq 2$, where $w_{ij}, w_{kl} \in V(G_1)$ (or $V(G_2)$), $i, k \in \{1, 2, 3\}, j, l \in \{1, 2, \ldots, n\}$. Thus, there is not any pair of vertices that is colored with color 3 or color 4. Then we have

$$\chi_{lc}(P_3 \times P_n) = 2 + \left\lceil \frac{n}{2} \right\rceil n + 2 = n + 4.$$

**Case 3.2.** Let $m = 5$ and $n \geq 6$.

If $\kappa(w_{ij}, w_{kl}) \geq 3$ or $\deg(w_{ij}) \geq 3$, where $i, k \in \{1, 2, \ldots, m\}, j, l \in \{1, 2, \ldots, n\}$, any two vertices $w_{ij}$ and $w_{kl}$ can be colored with color 3. Thus, there are $\left\lceil \frac{m}{2} \right\rceil (\frac{n}{2} - 1)$
vertices in $G_1$ each of degree 4 that can be colored with color 3. Since $m = 5$, only four vertices of them are colored with color 3.

Case 3.2.1. If $m = 5$ and $n = 6$, there is not any pair of vertices which receives color 4. Hence, the remaining vertices are colored with different colors, and so we have $\chi_{lc}(P_5 \times P_6) = 3 + \lfloor \frac{m}{2} \rfloor n + 2 - 4 = 13$.

Case 3.2.2. Let $m = 5$ and $n \geq 8$. The number of internally disjoint paths between only two vertices in $G_2$ is 4. Thus, these two vertices are colored with color 4. Since the remaining $2n - 4$ vertices receive different colors, we have $\chi_{lc}(P_5 \times P_n) = 2n$.

Case 3.3. Let $m \geq 7$ and $n \geq 8$.

Since $m \geq 7$, in $G_1$ there are $\lfloor \frac{m}{2} \rfloor (\frac{n}{2} - 1)$ vertices each of degree 4 which are colored with color 3. Thus, we have

$$\chi_{lc}(G_1) = 2 + \lfloor \frac{m}{2} \rfloor \frac{n}{2} - \lfloor \frac{m}{2} \rfloor (\frac{n}{2} - 1) = 2 + \lfloor \frac{m}{2} \rfloor.$$ 

Case 3.3.1. If $m = 7$ and $n \geq 8$, only four vertices in $G_2$ can be colored with color 4. Since the remaining $\frac{n}{2} \lfloor \frac{m}{2} \rfloor - 2$ vertices in $G_2$ receive different colors, $\chi_{lc}(G_2) = 2 + \frac{n}{2} \lfloor \frac{m}{2} \rfloor - 2 = \frac{mn}{2} - 2$, and we have $\chi_{lc}(P_7 \times P_n) = \chi_{lc}(G_1) + \chi_{lc}(G_2) = \frac{3n}{2} + 5$.

Case 3.3.2. Let $m \geq 9$ and $n \geq 10$.

For the minimum number of local connective coloring consider the vertices $w_{ij}$, where $i \in \{4, 6, \ldots, m - 3\}$, $j \in \{3, 5, \ldots, n - 1\}$ and $w_{kl}$, where $k \in \{2, m - 1\}$, $l \in \{5, 7, \ldots, n - 3\}$ in $G_2$. Since the number of internally disjoint paths between these $\frac{m - 4}{2} \lfloor \frac{n - 2}{2} \rfloor + 2 \frac{n - 2}{2}$ vertices is 4, they are colored with color 4. The remaining $\frac{mn}{2} - 11$ vertices in $G_2$ receive different colors. Hence, $\chi_{lc}(G_2) = 2 + \frac{mn}{2} - 11$ and we have $\chi_{lc}(P_m \times P_n) = \chi_{lc}(G_1) + \chi_{lc}(G_2) = m + 9$.

Consequently, if $m$ is odd and $n$ is even, then we get

$$\chi_{lc}(P_m \times P_n) = \begin{cases} 
\frac{n + 4}{2}, & m = 3, n \geq 4 \\
13, & m = 5, n = 6 \\
2n, & m = 5, n \geq 8 \\
\frac{3n}{2} + 5, & m = 7, n \geq 8 \\
m + 9, & m \geq 9, n \geq 10.
\end{cases}$$

Case 4. Let $m$ and $n$ be even.

In this case, $|V(G_1)| = |V(G_2)| = \frac{mn}{2}$. The graph $P_m \times P_n$ has four vertices of each of degree one. Two of them are in $G_1$ and the other two vertices are in $G_2$. Assume that we start coloring the graph from the vertex $w_{11}$ in $G_1$, and assign color 1 to every vertex $w_{ij}$ when $i$ and $j$ are not both even. Thus, $\beta(G_1) = \frac{mn}{4}$ vertices are colored with color 1 and $\frac{mn}{4}$ vertices in $G_1$ remain uncolored.

For color 2, assume that we start coloring the graph $G_2$ from the vertex $w_{21}$, and consider the vertices $w_{ij}$, where $i \in \{2, 4, \ldots, m - 2\}$, $j \in \{1, 3, \ldots, n - 1\}$ and $w_{nl}$, where $l \in \{3, 5, \ldots, n - 1\}$. Thus, $(\frac{m - 2}{2}) \lfloor \frac{n - 1}{2} \rfloor + \frac{n - 2}{2} = \frac{mn}{4} - 1$ vertices in $G_2$...
are colored with color 2, and the number of the remaining uncolored vertices in \( G_2 \) is \( \frac{mn}{4} + 1 \).

**Case 4.1.** Let \( m = 2 \) and \( n \geq 2 \).

Since \( P_2 \times P_n \cong 2P_n \) and \( \chi_{lc}(P_n) = 1 + \left\lceil \frac{n}{2} \right\rceil \) by [7], we have

\[
\chi_{lc}(P_2 \times P_n) = n + \chi_{lc}(P_n) = n + 1 + \left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{3n + 1}{2} \right\rceil.
\]

**Case 4.2.** Let \( m = 4 \) and \( n \geq 4 \).

Since \( m = 4 \), there is not any pair of vertices which is colored with color 3 and color 4. Hence, we have

\[
\chi_{lc}(P_4 \times P_n) = 2 + \frac{mn}{2} + 1 = 2n + 3.
\]

**Case 4.3.** Let \( m \geq 6 \) and \( n \geq 6 \).

For every vertex of degree 4 in \( G_1 \) which is not adjacent with each other \( \kappa(w_{ij}, w_{kl}) \geq 3 \) is satisfied, where \( i, k \in \{2, 4, \ldots, m - 2\} \), \( j, l \in \{2, 4, \ldots, n - 2\} \). Thus, \( (\frac{m - 2}{2})(\frac{n - 2}{2}) \) vertices are colored with color 3, and \( \frac{mn}{2} - 1 \) vertices in \( G_1 \) remain uncolored. Since the number of internally disjoint paths between these remaining vertices is at most 2, all of them receive different colors. Hence, we have

\[
\chi_{lc}(G_1) = 1 + \frac{mn}{4}.
\]

**Case 4.3.1.** Let \( m = n = 6 \). Since there is not any pair of vertices in \( G_2 \) which is colored with color 4, we have \( \chi_{lc}(P_6 \times P_6) = \chi_{lc}(G_1) + 1 + \frac{mn}{4} + 1 = 18 \).

**Case 4.3.2.** Let \( m = 6, n \geq 8 \). The number of internally disjoint paths between only two vertices in \( G_2 \) is 4. Thus, the remaining \( \frac{mn}{2} - 1 \) vertices receive different colors, and we have \( \chi_{lc}(P_6 \times P_n) = \chi_{lc}(G_1) + 2 + \frac{mn}{4} - 1 = 2n + 5 \).

**Case 4.3.3.** Let \( m \geq 8 \) and \( n \geq 8 \).

Consider the vertices \( w_{ij}, \) where \( i \in \{5, 7, \ldots, m - 3\}, j \in \{2, 4, \ldots, n - 2\} \) and \( w_{kl}, \) where \( k \in \{3, m - 1\}, l \in \{4, 6, \ldots, n - 4\} \) in \( G_2 \). The number of internally disjoint paths between these vertices is 4. Thus, \( \frac{m - 5}{2}(\frac{n - 2}{2}) + 2(\frac{n - 2}{2}) + \frac{mn}{8} - m + 3 \) vertices receive color 4. The remaining \( \frac{mn}{2} + 4 \) vertices receive different colors. Then \( \chi_{lc}(G_2) = 6 + \frac{mn}{4} \) and we have \( \chi_{lc}(P_m \times P_n) = \chi_{lc}(G_1) + \chi_{lc}(G_2) = m + n + 7 \).

As a result, if \( m \) and \( n \) is even, we have

\[
\chi_{lc}(P_m \times P_n) = \begin{cases} 
\left\lceil \frac{3n + 1}{2} \right\rceil, & m = 2, n \geq 2 \\
2n + 3, & m = 5, n \geq 4 \\
18, & m = n = 6 \\
2n + 5, & m = 6, n \geq 6 \\
m + n + 7, & m \geq 8, n \geq 8 .
\end{cases}
\]

By summing up four cases we have the statement of Theorem. \( \square \)

**Theorem 2.2.** Let \( C_m \) and \( P_n \) be cycle and path of order \( m \) and \( n \), respectively. Then,
\[
\chi_{lc}(C_m \times P_n) = \begin{cases} 
2, & \text{if } m \text{ is odd} \\
 m + 2, & \text{if } m \text{ is even, } n = 2 \\
 m + 3, & \text{if } m \text{ is even, } n = 3 \\
 m + 4, & \text{if } m \text{ is even, } n \geq 4 \text{ even} \\
or m \text{ is even, } n \geq 5 \text{ odd}
\end{cases}
\]

**Proof.** Since \(d(e_{ij}) = d(u_i) \cdot d(v_j)\), we have

\[\kappa(w_{ij}, w_{kl}) \leq \min\{w_{ij}, w_{kl}\} \leq 4,\]

where \(w_{ij}, w_{kl} \in V(C_m \times P_n)\), \(i, k \in \{1, 2, \ldots, m\}, j, l \in \{1, 2, \ldots, n\}\). Thus, \(k_i \leq 1\) for \(i \geq 5\), and the pair of vertices can be colored with the same color at most color 4. We have following two cases for coloring the graph \(C_m \times P_n\).

**Case 1.** Let \(m\) and \(n\) be odd or \(m\) be odd and \(n\) be even.

By Theorem 1.1, since \(m\) is odd, the graph \(G = (V_1 \cup V_2, E)\). Since \(\kappa(w_{ij}, w_{kl}) \leq 4\), where \(w_{ij}, w_{kl} \in V(C_m \times P_n)\), \(i, k \in \{1, 2, \ldots, m\}, j, l \in \{1, 2, \ldots, n\}\), the vertices of \(V_1\) and \(V_2\) can be colored with color 1 and color 2, respectively. Then we have \(\chi_{lc}(C_m \times P_n) = 2\).

**Case 2.** Let \(m\) and \(n\) be even or \(m\) be even and \(n\) be odd.

In this case, since cycles and paths are bipartite graphs, let \(G_1 = (V_0 \cup V_1, E), G_2 = (V_0 \cup V_1, E)\). By Theorem 1.3 and Theorem 1.2, the graph \(G_1\) is non-connected bipartite graph. Further, by Lemma 1.2 and 1.3, \(G_1 = (V_0 \times W_0) \cup (V_1 \times W_1), G_2 = (V_0 \times W_1) \cup (V_1 \times W_0), E_{G_1}, E_{G_2}\) are two bipartite components of \(G_1\). Since \(G_1\) is bipartite graph, \(\chi_{lc}(G_1) = 2\) by Case 1. Let’s start coloring the graph \(G_2\) with color \(\chi_{lc}(G_1) + 1 = 3\).
Since $|V(C_m \times P_n)| = |V(C_m)||V(P_n)|$, it is obvious that $|V(C_n)| = |V(P_m)| = \frac{mn}{2}$.

**Case 2.1.** Let $n = 2$ and $m$ be even.

Since $P_2 = K_2$, by Lemma 1.1 the graph $C_m \times K_2$ has exactly two connected components $G_1$ and $G_2$ isomorphic to $C_m$.

Since $G_2$ is 2–regular graph, the number of internally disjoint paths between all two vertices in $G_2$ is at most 2. Thus, all vertices of $G_2$ receive different colors. Then, $\chi_{lc}(C_m \times P_2) = \chi_{lc}(G_1) + \chi_{lc}(G_2) = m + 2$.

**Case 2.2.** Let $n = 3$ and $m$ be even.

In this case, the graph $G_2$ has $\frac{m(n-2)}{2} = \frac{m}{2}$ vertices of degree 4, and these vertices are either in the vertex set $V_0 \times W_1$ or in $V_1 \times W_0$. That is, these vertices are not adjacent. Thus, $\frac{m}{2}$ vertices are colored with color 3. The number of the remaining uncolored vertices is $m$. Since the degree of these remaining vertices is 2, these $m$ vertices receive different colors. Thus, we have

$$\chi_{lc}(C_m \times P_3) = \chi_{lc}(G_1) + \chi_{lc}(G_2) = 2 + 1 + m = m + 3.$$ 

**Case 2.3.** Let $m$ be even and $n \geq 4$ even or $m$ be even and $n \geq 5$ odd.

The graph $C_m \times P_n$ has $m(n-2)$ vertices each of degree 4, and $G_2$ has half of these vertices. The vertex sets $W_0$ and $W_1$ have $\lfloor \frac{n-2}{2} \rfloor$ and $\lceil \frac{n-2}{2} \rceil$ internal vertices of $P_n$, respectively.

Since $\kappa(w_{ij}, w_{kl}) \leq 4$, where $w_{ij}, w_{kl} \in V(G_2)$, we color $\frac{m}{2} \lfloor \frac{n-2}{2} \rfloor$ vertices of $G_2$ with color 3 and $\frac{m}{2} \lceil \frac{n-2}{2} \rceil$ vertices of $G_2$ with color 4. The remaining $m$ vertices receive different colors. Thus, we have

$$\chi_{lc}(C_m \times P_n) = \chi_{lc}(G_1) + \chi_{lc}(G_2) = 2 + 2 + m = m + 4.$$ 

**Theorem 2.3.** Let $C_m$ and $C_n$ be two cycles of order $m$ and $n$, respectively. Then,

$$\chi_{lc}(C_m \times C_n) = \begin{cases} 
2, & \text{if } m \text{ is odd, } n \text{ is even or } m \text{ is even, } n \text{ is odd} \\
3, & \text{if } m \text{ and } n \text{ are odd} \\
4, & \text{if } m \text{ and } n \text{ are even.}
\end{cases}$$

**Proof.** By proof of Theorem 1.4, the graph $C_m \times C_n$ is 4–regular graph. Then the number of internally disjoint paths between any two vertices in $C_m \times C_n$ is at most 4. Hence, $k_i \leq 1$ for $i \geq 5$. We have following three cases for coloring the graph $C_m \times C_n$.

**Case 1.** Let $m$ be odd and $n$ be even or $m$ be even and $n$ be odd.

It is known that if the order of a cycle is even, it is bipartite graph. Thus, by Theorem 1.1 and Theorem 1.2, the graph $C_m \times C_n$ is bipartite connected graph. Since the number of internally disjoint paths between any two vertices in $C_m \times C_n$ is at most 4, we have $\chi_{lc}(C_m \times C_n) = 2$.

**Case 2.** Let $m$ and $n$ be even.

Since $m$ and $n$ are even, $C_m$ and $C_n$ are bipartite graphs. By Theorem 1.2, Theorem 1.3 and Lemma 1.3, the graph $C_m \times C_n$ is bipartite non-connected graph.
and has exactly two isomorphic connected components $G_1$ and $G_2$. Thus, these components are also bipartite graphs. Since $G_1$ is bipartite graph and the number of internally disjoint paths between any two vertices of $G_1$ is at most 4, we have $\chi_{lc}(G_1) = 2$. Let’s start coloring the graph $G_2$ with color $\chi_{lc}(G_1) + 1 = 3$. Since $G_2$ is also bipartite graph and the number of internally disjoint paths between any two vertices of $G_2$ is at most 4, the vertices of $G_2$ receive color 3 and color 4. Hence, $\chi_{lc}(C_m \times C_n) = 4$.

**Case 3.** Let $m$ and $n$ be odd.

Since $m$ and $n$ are odd, the graph $C_m \times C_n$ is connected by Theorem 1.1. Assume that we start coloring the graph from the vertex $w_{11}$. By definition of direct product, the vertex $w_{1j}$ can be colored with color 1 for $j \in \{1, 2, ..., n\}$. Thus we color all non-adjacent vertices $w_{1j}$, $w_{3j}$, $w_{5j}$, ..., $w_{(m-2)j}$ with color 1. Since the vertices $w_{2j}$, $w_{4j}$, $w_{6j}$, ..., $w_{(m-1)j}$ are not adjacent with each other, and the number of internally disjoint paths between them is at most 4, these vertices receive color 2. Then total $n(m - 1)$ vertices in $C_m \times C_n$ are colored with color 1 and color 2. Hence, the vertex $w_{mj}$ remains uncolored and the number of its vertices is $mn - n(m - 1) = n$. Since the vertex $w_{mj}$ is adjacent to the vertices $w_{1j}$ and $w_{(m-1)j}$, the vertex $w_{mj}$ is colored with different color other than color 1 and color 2. Thus, all vertices in the graph $C_m \times C_n$ are colored with three local connective colors.

3. Conclusion

In this paper, we define a new type of graph coloring called local connective coloring. It is known that a communication network is fault-tolerant if it has alternative paths (internally disjoint paths) between vertices and the internally disjoint paths are used to transmit messages among vertices. Thus, we use the
term internally disjoint path in our coloring and color the vertices depending on the number of internally disjoint paths between two vertices. In our work, we study on the local connective chromatic number of direct product of some cycles and paths. We can consider the local connective chromatic number of Cartesian product of graphs in further study.

References


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