ir-EXCELLENT GRAPHS

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Abstract. Teresa W. Haynes et. al. [7], introduced the concept of irredun-
dance in graphs. A subset $S$ of $V(G)$ is called an irredundant set of $G$ if for
every vertex $u \in S$, $pn(u, S) \neq \emptyset$. The minimum (maximum)cardinality of a
maximal irredundant set of $G$ is called the irredundance number of $G$ (upper
irredundance number of $G$) and is denoted by $ir(G)(IR(G))$. A subset $V(G)$
is called an ir-set if it is an irredundant set of $G$ of cardinality $ir(G)$. A vertex
$u \in V(G)$ is called ir-good if $u$ belongs to an ir-set of $G$. $G$ is said to be ir-
excellent if every vertex of $G$ is ir-good. In this paper, a study of the excellent
graphs with respect to irredundance is initiated.

1. Introduction

We consider the graphs which are finite, undirected, non - trivial without loops
or multiple edges. Let $G = (V, E)$ be a simple graph. For graph theoretic termi-
nology, we refer to [1]. A subset $S$ of $V$ is a dominating set of $G$ if every vertex in
$V - S$ is adjacent to some vertex in $S$. The domination number $\gamma(G)$ of $G$ is the
minimum cardinality of a dominating set of $G$. For a set $S$ of vertices in a graph
$G$, the closed neighborhood $N[S]$ of $S$ is defined $N[S] = \bigcup_{v \in S} N[v]$. Each vertex
in $N[v]N[Sv]$ is referred to as a private neighbour of $v \in G$ and is denoted by
$pn(v, S)$. In [7], a subset $S$ of $V(G)$ is called an ir-set if it is an irredundant set of cardinality $ir(G)$ ($ir(G)$ is the minimum cardinality of a maximal irredundant set). Any non empty subset of an irredundant set is irredundant. Hence, the property
of irredundance is hereditary.

Let $\mu$ be a parameter of a graph. A vertex $v \in V(G)$ is said to be $\mu$-good if $v$
belongs to a $\mu$-minimum ($\mu$-maximum) set of $G$ according as $\mu$ is a super hereditary
(hereditary) parameter. $v$ is said to be -bad if it is not $\mu$-good. A graph $G$ is said to

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be $\mu$-excellent if every vertex of $G$ is $\mu$-good. Excellence with respect to domination and total domination were studied in [2]. In a social network, we may exchange any node inside the network by a node outside the network, gives a better status in the form of a new group. Such a situation can be modelled as a set $S$ of vertices in the graph $G$ representing the social network such that for every $y \in V(G) - S$ there exists $x \in S$ such that the new social group $S = (S - \{x\}) \cup \{y\}$ has the same property as that of $S$ and is possibly better in terms of external connections as well as its internal organization. This is the motivation for studying excellent graphs with various graph parameters. N. Sridharan and Yamuna [4, 5, 6], have defined various types of excellence.

2. ir-excellent graphs

In this section, we define and study a new type of graph, namely ir-excellent graph.

**Definition 2.1.** Let $G = (V, E)$ be a simple graph. Then $G$ is said to be an ir-excellent graph if every vertex belongs to an ir-set of $G$.

**Example 2.1.** ir-excellent graphs.

1. $K_n$
2. $\overline{K_n}$
3. $C_n$
4. $K_{n,n}$, $n \geq 2$
5. $K_{m,n}$, $m, n \geq 2$, $m < n$
6. $D_{r,s}$ is ir-excellent if $r = s = 1$.

**Example 2.2.** Graphs which are not ir-excellent

1. $K_{1,n}$
2. $D_{r,s}$ for $r, s \geq 2$ (ir($D_{r,s}$) = 2, IR($D_{r,s}$) = $r + s$).

Let $V(D_{r,s}) = \{u_1, u_2, \ldots, u_r, u, v, v_1, v_2, v_s\}$ where $u$ is the support of the pendent vertices $u_1, u_2, \ldots, u_r$ and $v$ is the support of the pendent vertices $\{v_1, v_2, \ldots, v_s\}$.

Let $S = \{u, v\}$. Then $S$ the only ir-set of $D_{r,s}$, since all the pendent vertices are not in any ir-set of $D_{r,s}$.

**Proposition 2.1.** If $G$ is vertex transitive then $G$ is ir-excellent.

**Proof.** Let $D$ be an ir-set of $G$. Let $u \notin D$. Let $v \in D$. Then there exists an automorphism $\phi$ such that $\phi(v) = u$. Then $u \in \phi(D)$.

**Claim 1:** $\phi(D)$ is irredundant.

For: Let $w \in \phi(D)$. Then $w = \phi(y)$ for some $y \in D$. If $y$ is the private neighbourhood of itself with respect to $D$, then $y$ is an isolate of $D$, which implies $\phi(y)$ is an isolate of $\phi(D)$. Therefore $w$ is a private neighbourhood of itself with respect to $\phi(D)$. If $y_1$ is a private neighbourhood of $y$ with respect to $D$, then $y_1$ is not adjacent to any vertex of $D$ other than $y$. Therefore $\phi(y_1)$ is not adjacent to any vertex of $\phi(D)$ other than $\phi(y) = w$. Hence $w$ has a private neighbourhood $\phi(y)$ with respect to $\phi(D)$. Therefore $\phi(D)$ is irredundant.
Claim 2: $\phi(D)$ is a maximal irredundant set.

Suppose not. Then there exists $S \subset V(G)$ such that $\phi(D) \subsetneq S$ and $S$ is irredundant. Let $x \in S - \phi(D)$. Let $\phi^{-1}(x) = t$. Then $t \in \phi^{-1}(S)$ and $t \notin D$. Therefore $D \subsetneq \phi^{-1}(S)$ and $\phi^{-1}(S)$ is irredundant, a contradiction to maximality of $D$. Therefore $\phi(D)$ is a maximal irredundant set of $G$. $|D| = |\phi(D)| = ir(G)$ and hence $\phi(D)$ is an $ir$-set of $G$ containing $u$. Therefore $u$ is $ir$-good and $G$ is $ir$-excellent. \hfill $\Box$

Observation 2.1. Let $\gamma(G) = ir(G)$. If $G$ is $\gamma$-excellent, then $G$ is $ir$-excellent.

Observation 2.2. There exists a graph $G$ in which $\gamma(G) = ir(G)$, $G$ is $ir$-excellent but not $\gamma$-excellent.

Example 2.3.

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) [circle,fill,inner sep=2pt]{1};
\node (2) at (-1,-1) [circle,fill,inner sep=2pt]{2};
\node (3) at (0,-1) [circle,fill,inner sep=2pt]{3};
\node (4) at (1,-1) [circle,fill,inner sep=2pt]{4};
\node (5) at (-1,-2) [circle,fill,inner sep=2pt]{5};
\node (6) at (0,-2) [circle,fill,inner sep=2pt]{6};
\node (7) at (1,-2) [circle,fill,inner sep=2pt]{7};
\draw (1) -- (2);
\draw (1) -- (3);
\draw (1) -- (4);
\draw (2) -- (5);
\draw (2) -- (6);
\draw (3) -- (6);
\draw (4) -- (7);
\end{tikzpicture}
\end{center}

$\gamma$-sets of $G$ are: $\{2,3,4\}, \{2,4,6\}, \{2,6,7\}, \{4,5,6\}, \{5,3,7\}$

$ir$-sets of $G$ are: $\{1,3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,5,7\}, \{1,5,6\}, \{1,6,7\}$,
$\{3,5,7\}, \{2,3,4\}, \{2,3,7\}, \{3,4,5\}, \{2,4,6\}, \{5,6,7\}$

1 does not belong to any $\gamma$-set. Therefore $G$ is not $\gamma$-excellent. But $G$ is $ir$-excellent.

Observation 2.3. There exists a graph $G$ in which $ir(G) < \gamma(G)$, $G$ is not $ir$-excellent but $\gamma$-excellent.

Consider the Allan Laskar graph (A. L. graph), which is shown below:

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) [circle,fill,inner sep=2pt]{1};
\node (2) at (-1,-1) [circle,fill,inner sep=2pt]{2};
\node (3) at (-1,-2) [circle,fill,inner sep=2pt]{3};
\node (4) at (0,-2) [circle,fill,inner sep=2pt]{4};
\node (5) at (1,-2) [circle,fill,inner sep=2pt]{5};
\node (6) at (1,-1) [circle,fill,inner sep=2pt]{6};
\node (7) at (2,0) [circle,fill,inner sep=2pt]{7};
\draw (1) -- (2);
\draw (1) -- (3);
\draw (1) -- (4);
\draw (2) -- (3);
\draw (2) -- (4);
\draw (2) -- (5);
\draw (3) -- (4);
\draw (3) -- (5);
\draw (4) -- (7);
\end{tikzpicture}
\end{center}

$ir$-set: $\{3,5\}$
\(\gamma\)-sets: \(\{1,3,7\}\), \(\{2,4,6\}\), \(\{5,2,7\}\).

The graph is \(\gamma\)-excellent but not \(\text{ir}\)-excellent.

In general the above type of graphs with a subgraph as a complete graph have the property \(\text{ir} < \gamma\), \(\text{ir} = 2\) and \(\gamma = 3\). Such type of graphs are \(\gamma\)-excellent graphs but not \(\text{ir}\)-excellent.

**Proposition 2.2.** For any path \(P_n\), \(\text{ir}(P_n) = \gamma(P_n)\).

**Proof.** \(\Delta(P_n) = 2\). From [1], we have \(\frac{2n}{2\Delta(P_n) - 1} \leq \text{ir}(G)\). Therefore \(\frac{2n}{2\Delta(P_n) - 1} \leq \text{ir}(P_n)\), that is \(\frac{n}{3} \leq \text{ir}(P_n)\). Therefore \(\left\lfloor \frac{n}{3} \right\rfloor \leq \text{ir}(P_n)\), which means \(\gamma(P_n) \leq \text{ir}(P_n)\). But \(\text{ir}(P_n) \leq \gamma(P_n)\). Therefore \(\gamma(P_n) = \text{ir}(P_n)\). \(\Box\)

**Proposition 2.3.**

(1) \(P_{3n+1}\) is \(\text{ir}\)-excellent for all \(n\).

(2) \(P_{3n+2}\) is not \(\text{ir}\)-excellent for \(n \geq 3\).

(3) \(P_{3n}\) is not \(\text{ir}\)-excellent for all \(n\).

**Proof.** (1). Let \(V(P_{3n+1}) = \{u_1, u_2, \ldots, u_{3n+1}\}\). \(\gamma(P_{3n+1}) = \text{ir}(P_{3n+1}) = n + 1\).

\[D_1 = \{u_1, u_4, u_7, \ldots, u_{3n+1}\}, D_2 = \{u_2, u_5, \ldots, u_{3n-1}, u_3\text{ or } u_{3n+1}\}\]

are minimum dominating sets and hence also \(\text{ir}\)-sets of \(P_{3n+1}\). Therefore \(P_{3n+1}\) is \(\text{ir}\)-excellent.

(2). Let \(V(P_{3n+2}) = \{u_1, u_2, \ldots, u_{3n+2}\}\), \(n \geq 3\). \(\gamma(P_{3n+2}) = \text{ir}(P_{3n+2}) = n + 1\).

\[D_1 = \{u_1, u_4, \ldots, u_{3n+1}\}, D_2 = \{u_2, u_5, u_8, \ldots, u_{3n+2}\}, \]
\[D_3 = \{u_3, u_4, u_7, \ldots, u_{3n+1}\}, D_4 = \{u_2, u_5, u_8, \ldots, u_{3n-1}, u_{3n}\}\]

are all \(\text{ir}\)-sets and hence \(u_i\), \(i \neq 6, 9, \ldots, 3n - 3\) are not \(\text{ir}\)-good.

When \(n = 2\), both \(u_3\) and \(u_6\) will be \(\text{ir}\)-good and hence \(P_5\) is \(\text{ir}\)-excellent.

When \(n = 1\), \(u_3\) is \(\text{ir}\)-good and hence \(P_5\) is \(\text{ir}\)-excellent. Therefore \(P_{3n+2}\) is not \(\text{ir}\)-excellent if \(n \geq 3\).

(3). Let \(V(P_{3n}) = \{u_1, u_2, \ldots, u_{3n}\}\).

When \(n = 1\), \(P_3\) is not \(\text{ir}\)-excellent since \(\text{ir}(P_3) = 1\) and \(u_1\) is not in any \(\text{ir}\)-set.

When \(n = 2\), we get \(P_6\). Again \(u_1\) is not in any \(\text{ir}\)-set, since the minimum cardinality of an irredundant set containing \(u_1\) is \(3\) and \(\text{ir}(P_6) = 2\). \(\text{ir}(P_{3n}) = 2\) and the minimum cardinality of an irredundant set containing \(u_1\) is \(n + 1\). \([\{u_1, u_3, u_6, \ldots, u_{3n}\}\) or \(\{u_1, u_4, u_7, \ldots, u_{3n-2}, u_{3n}\}\) are irredundant sets containing \(u_1\) of minimum cardinality. Therefore \(P_{3n}\) is not \(\text{ir}\)-excellent. \(\Box\)

**Proposition 2.4.** If \(\text{ir}(G) < \gamma(G)\), then any independent set is not an \(\text{ir}\)-set.

**Proof.** Let \(S\) be an independent set of \(G\). Suppose \(S\) is an \(\text{ir}\)-set of \(G\). Then \(S\) is a maximal independent set of \(G\). Therefore \(S\) is a minimal dominating set of \(G\). Therefore \(\text{ir}(G) < \gamma(G) \leq |S| = \text{ir}(G)\), a contradiction. Therefore \(S\) is not an \(\text{ir}\)-set of \(G\). \(\Box\)
Corollary 2.1. If $ir(G) < \gamma(G)$, then for any ir-set $S$ of $G$, number of private neighbours of $S$ lying in $V - S$ is greater than or equal to 2.

Proposition 2.5. For any graph $G$, $G^+$ is both ir-excellent and $\gamma$-excellent.

Proof. Let $S$ be an ir-set of $G^+$. Suppose $|S| < n$. Then there exists $u \in V(G)$ such that $u, u' \notin S$ where $u'$ is the pendant of $u$. Then $S \cup \{u\}$ is an irredundant set of $G^+$, since $u$ is the private neighbour of $u$, a contradiction. Therefore $|S| \geq n$. Since $\gamma(G^+) = n$, $|S| = n$. Since any $\gamma$-set of $G^+$ is also an ir-set of $G^+$, $G^+$ is ir-excellent.

Observation 2.4. Any graph $G$ is an induced graph of an ir-excellent graph.

Proposition 2.6. Let $G$ be a non-ir-excellent graph with a unique ir-bad vertex. Then there exists an ir-excellent graph $H$ such that

(i). $G$ is an induced subgraph of $H$.

(ii). $ir(H) = ir(G) + 1$.

Proof. Let $u$ be the unique ir-bad vertex of $G$. Let $H$ be the graph obtained from $G$ by adding a new vertex $v$ and making it adjacent with only $u$ in $G$.

Claim: $ir(H) = ir(G) + 1$.

Let $ir(G) = k$. Note that for any ir-set $S$ of $G$, $u \notin S$. Hence $S \cup \{v\}$ is an irredundant set of $H$. Clearly it is a maximal irredundant set of $H$.

Suppose $ir(H) = k' \leq k$. Let $T$ be an ir-set of $H$. If $v \notin T$, then $T \cup \{v\}$ is an irredundant set of $H$ if $u \notin T$, a contradiction. Since $T$ is a maximal irredundant set of $H$, $u \in T$. Since $T \subseteq V(G)$, $T$ is a maximal irredundant set of $G$ and $k' = ir(G) \leq |T| = k' \leq k$. Therefore $|T| = k$. $T$ is an irredundant set of $G$ containing $u$, a contradiction, since $u$ is an ir-bad vertex of $G$. Therefore $v \in T$. Let $T_1 = T - \{v\}$. Then $T_1 \subseteq V(G)$. $|T_1| = k' - 1 < k$. Clearly $T_1$ being a subset of an irredundant set of $H$ is irredundant in $H$.

Case 1: $u \notin T_1$. Then $T_1$ is an irredundant set of $G$. Suppose $T_1$ is a maximal irredundant set of $G$. Then $k = ir(G) \leq |T_1| < k$, a contradiction. Therefore $T_1$ is not a maximal irredundant set of $G$. Therefore there exists $w \in G$ such that $T_1 \cup \{w\}$ is an irredundant set of $G$. Suppose $w \neq u$. Then $T \cup \{w\} = T_1 \cup \{w\} \cup \{v\}$ is an irredundant set of $H$ contradicting the maximality of $T$. Therefore $w = u$. Therefore $T_1 \cup \{u\}$ is an irredundant set of $G$. If $T_1 \cup \{u\}$ is a maximal irredundant set of $G$, then $k = ir(G) \leq |T_1| + 1 = k' \leq k$. Therefore $|T_1| + 1 = k$ and hence $T_1 \cup \{u\}$ is an ir-set of $G$ implying $u$ is ir-good, a contradiction. Therefore $T_1 \cup \{u\}$ is not a maximal irredundant set of $G$. Thus there exists $z \in V(G) - (T_1 \cup \{u\})$ such that $T_1 \cup \{u\} \cup \{z\}$ is an irredundant set of $G$. Therefore $T_1 \cup \{z\}$ is an irredundant set of $G$. Therefore $T_1 \cup \{z\} \cup \{v\}$ is an irredundant set of $H$. Thus $T \cup \{z\}$ is an irredundant set of $H$.

Case 2: $u \in T_1$. Then $T_1$ is an irredundant set of $G$. If $T_1$ is maximal, then $k = ir(G) \leq |T_1| < k$, a contradiction. Therefore $T_1$ is not a maximal irredundant set of $G$. Therefore there exists $x \in V(G) - T_1$ such that $T_1 \cup \{x\}$ is irredundant in $G$. Since $u \in T_1$, we get that $x \neq u$. Therefore $T_1 \cup \{x\} \cup \{v\}$ is an irredundant
set in \( H \). That is \( T \cup \{x\} \) is irredundant in \( H \), a contradiction to the maximality of \( T \). Therefore \( ir(H) > k \). That is \( ir(H) \geq k + 1 \). But \( S \cup \{v\} \) for any ir-set \( S \) of \( G \) is a maximal irredundant set of \( H \). Therefore \( ir(H) \leq |S \cup \{v\}| = k + 1 \). Therefore \( ir(H) = k + 1 \). Therefore \( S \cup \{v\} \) is an ir-set of \( H \) for any ir-set \( S \) of \( G \). Therefore every ir-good vertex in \( G \) as well as \( v \) is ir-good in \( H \). Moreover for any ir-set \( S \) of \( G \), \( S \cup \{u\} \) is irredundant in \( H \) since \( u \) has a private neighbour \( v \) in \( H \). Therefore \( S \cup \{u\} \) is an ir-set of \( H \), which implies \( u \) is also ir-good in \( H \). Therefore \( H \) is ir-excellent. \( G \) is an induced subgraph of \( H \). Further, \( ir(H) = ir(G) + 1 \). □

**Conjecture.** There does not exist any graph \( G \) which is both \( \gamma \)-excellent and ir-excellent and \( ir(G) < \gamma(G) \).

**Corollary 2.2.** If \( G_1 \), \( G_2 \) are ir-excellent, then \( G_1 + G_2 \) is ir-excellent if and only if \( ir(G_1) = ir(G_2) \).

## 3. Definition and Properties of just ir-excellent graphs

In this section, we introduce the concept of just ir-excellent graphs and study its properties.

**Definition 3.1.** A graph \( G \) is said to be just ir-excellent graph, if every vertex of \( G \) belongs to exactly one ir-set of \( G \).

**Remark 3.1.** If \( G \) is just ir-excellent then \( G \) admits a partition where each element of the partition is an ir-set of \( G \).

**Example 3.1.**
\( C_{3n}, K_n, H_{5,10} \).

**Remark 3.2.** Every just ir-excellent graph is ir-excellent graph.

**Remark 3.3.** If \( \gamma(G) = 2 \), then \( ir(G) = 2 \).

**Proof.** Suppose \( ir(G) = 1 \). Then \( G \) has a full degree vertex. Hence \( \gamma(G) = 1 \), a contradiction. Therefore \( ir(G) \geq 2 \). But \( ir(G) \leq \gamma(G) = 2 \). Therefore \( ir(G) = 2 \). The converse is not true, since in A.L graph \( \gamma(G) = 3 \) and \( ir(G) = 2 \). □

**Proposition 3.1.** It has been proved in [3] that if \( G \) is a graph containing no induced subgraph isomorphic to \( K_{1,3} \) or A.L graph, then \( ir(G) = \gamma(G) = i(G) \). Since \( C_n \) and \( P_n \) does not contain \( K_{1,3} \) or A.L graph as an induced subgraph, \( ir(C_n) = \gamma(C_n) = i(C_n) \) and \( ir(P_n) = \gamma(P_n) = i(P_n) \).

**Observation 3.1.** \( C_n \) is \( \gamma \)-excellent if and only if \( n \equiv 0 (mod 3) \). Therefore \( C_n \) is ir-excellent if and only if \( n \equiv 0 (mod 3) \).

**Proposition 3.2.** Every just ir-excellent graph \( G(\neq K_n) \), is connected.

**Proof.** Let \( G \) be a disconnected graph, \( G \neq K_n \). Let \( G_1 \) be a component of \( G \). If \( |V(G_1)| = 1 \), then \( G \) is not just ir-excellent. Hence \( |V(G_1)| \geq 2 \).

**Claim:** \( G_1 \) is just ir-excellent.
Let $S$ be an ir-set of $G$. Let $S_1 = S \cap V(G_1)$. Clearly, $S_1$ is non-empty. Since $S$ is an ir-set, $S_1$ is an irredundant set of $G_1$ and clearly it is a maximal irredundant set of $G_1$.

Suppose $|S_1| > \mathrm{ir}(G)$. Let $S'$ be an ir-set of $G$. Then $S' \cup (S - S_1)$ is an irredundant set of $G$ of cardinality greater than $|S|$, a contradiction (since $S$ is an ir-set of $G$). Therefore $S_1$ is an ir-set of $G_1$. Since $G$ is just ir-excellent, $G_1$ is also just ir-excellent. Since $G_1$ is connected, $\mathrm{ir}(G_1) \leq \gamma(G_1) \leq \frac{2}{3}$. As $G_1$ is just ir-excellent, $G_1$ has at least two ir-sets, say $T_1$ and $T_2$. Let $D_1$ be an ir-set of $G - G_1$. Then $D_1 \neq \emptyset$ and $T_1 \cup D_1$, $T_2 \cup D_2$ are ir-sets of $G$ with non-empty intersection, a contradiction, since $G$ is just ir-excellent. Therefore $G$ is connected. \hfill $\square$

**Proposition 3.3.** Let $G \neq \overline{K_n}$ is just ir-excellent. Then for any ir-set $D$ of $G$, $|\mathrm{pm}[u, D]| \geq 2$ for all $u \in D$.

**Proof.** **Case A:** Since $G \neq \overline{K_n}$, order of $G$ is greater than or equal to 2.

Since $D$ is an ir-set of $G$, $|\mathrm{pm}[u, D]| \geq 1$ for all $u \in D$. Suppose $|\mathrm{pm}[u, D]| = 1$.

**Case (i):** $|\mathrm{pm}[u, D]| = 1$. Let $\mathrm{pm}(u, D) = \{v\}$ where $v \in V - D$. Let $D_1 = (D - \{u\}) \cup \{v\}$. Then $v$ being not adjacent to any vertex of $D - \{u\}$, $v \in \mathrm{pm}(v, \{D - \{u\}\} \cup \{v\})$. Also, if $x \in D - \{u\}$, then $\mathrm{pm}(x, D) = \mathrm{pm}(x, \{D - \{u\}\} \cup \{v\})$, since $v$ is not adjacent with $x$. Therefore $D_1$ is an irredundant set of $G$ of cardinality $\mathrm{ir}(G)$.

Suppose $D_1$ is not a maximal irredundant set of $G$. Then there exists a maximal irredundant set say $D_2$ of $G$ such that $D_1 \subseteq D_2$. Let $w \in D_2 - D_1$.

**Subcase (i):** $w = u$. In this case $D_1 \subseteq D_2$ and $v \in D_2$. Since $u$ and $v$ are adjacent and $D_2$ is irredundant, $w = u$ has a private neighbour say $x$ with respect to $D_2$ outside $D_2$. Clearly $x \notin D$. Therefore $x$ and $v$ are two private neighbours of $u$ with respect to $D$ belonging to $V - D$, a contradiction since $|\mathrm{pm}(u, D)| = 1$.

**Subcase (ii):** $w \neq u$. Clearly $w \neq v$. Since $u$ is adjacent with $v \in D_2$, $u$ cannot be a private neighbour of $w$ with respect to $D_2$. Therefore $w$ is a private neighbour of $u$ with respect to $D$. Hence $|\mathrm{pm}(u, D)| \geq 2$, a contradiction.

**Subcase (iii):** Suppose $u$ is not a private neighbour of $w$ with respect to $D_2$. Then either $w$ is an isolate of $D_2$ or there exists $y \in V - D_2$ such that $y \in \mathrm{pm}(w, D_2)$. Let $w$ be an isolate of $D_2$. Consider $D' = D \cup \{w\}$. If $w$ is not adjacent with $u$ then $w$ is an isolate of $D'$ and hence $D'$ is an irredundant set containing $D$, a contradiction to maximality of $D$. If $w$ is adjacent with $u$, then $w$ being not adjacent with any vertex of $D - \{u\}$, is a private neighbour of $u$ with respect to $D$ in $V - D$. That is $u$ has two private neighbours $v, w$ with respect to $D$ in $V - D$, a contradiction since $|\mathrm{pm}(u, D)| = 1$. Suppose there exists $y \in V - D_2$ such that $y \in \mathrm{pm}(w, D_2)$. Let $w$ be a private neighbour of some $x \in D$ with respect to $D$. If $x = u$, then $u$ has two private neighbours with respect to $D$, a contradiction. If $x \neq u$, then as $x, w \in D_2$, $x$ has a private neighbour say $z$ outside $D_2$ with respect to $D_2$. Then $z$ is a private neighbour of $x$ with respect to $D$. Hence $D \cup \{w\}$ is an irredundant set of $G$, containing $D$, a contradiction to the maximality of $D$.

**Case (ii):** $u$ is an isolate of $D$. Since $u$ is not an isolate of $G$ (if $u$ is an isolate of $G$, then $u$ belongs to every irredundant set contradicting just ir-excellent), there
exists $v \in V - D$ such that $u$ and $v$ are adjacent. Since $pn[u, D] = \{ u \}$, $v$ is not a private neighbour of $u$ with respect to $D$. Therefore $v$ is adjacent to some vertex say $w \neq u \in D$. Consider $D_1 = (D - \{ u \}) \cup \{ v \}$. Since $u$ is a private neighbour of $v$ with respect to $D_1$ and since every vertex of $D - \{ u \}$ has a private neighbour not equal to $v$ with respect to $D$, $D_1$ is an irredundant set of $G$ strictly containing $D_2$. Let $w \in D_2 - D_1$. Suppose $w = u$. Since $u$ is adjacent with $v$, $w$ in $D_2$ is not an isolate of $D_2$. $u = w$ has a private neighbour in $V - D$ with respect to $D$. Therefore $|pn[u, D]| \geq 2$, a contradiction.

Suppose $w \neq u$.

Subcase (i): $w$ is an isolate of $D_2$. Then $w$ is not adjacent with any vertex of $(D - \{ u \}) \cup \{ v \}$. (If $w$ is adjacent with $u$ then $w$ is a private neighbour of $u$ in $V - D$ with respect to $D$ a contradiction). Therefore $w$ is not adjacent with $u$. $w$ is an isolate of $D$ with respect to $w$. Hence $D \cup \{ w \}$ is an irredundant set containing $D$, a contradiction to the maximality of $D$.

Subcase (ii): $w$ is not an isolate of $D_2$. Then $w$ has a private neighbour say $z$ in $V - D_2$. If $z = u$, then $z$ is not adjacent with any vertex in $D_2 - w$. But $u$ is adjacent with $v$ in $D_2$, a contradiction. Therefore $z \neq u$. Consider $D \cup \{ w \}$. If $w$ is not a private neighbour of any vertex of $D$ with respect to $D$, then $D \cup \{ w \}$ is irredundant. If $w$ is a private neighbour of some $x \in D$ with respect to $D$, then $x \neq u$ (since $pn[u, D] = 1$). As $x$ and $w$ are adjacent in $D_2$, $x$ has a private neighbour say $y$ in $V - D_2$ with respect to $D_2$. That is $x$ has a private neighbour $y$ in $V - D$ with respect to $D$. Therefore $D \cup \{ w \}$ is an irredundant set of $G$ is a contradiction to the maximality of $D$. Therefore $D_1 = (D - \{ u \}) \cup \{ v \}$ is a maximal irredundant set of $G$. $|D| = 1$ implies $ir(G) = 1$. As $G$ is just excellent and $ir(G) = 1$, $G = K_n$, a contradiction. Therefore $|D| \geq 2$. Hence $\phi \neq D - \{ u \}$, is contained in two $ir$-sets namely $D$ and $D_1$, a contradiction to just excellence. Therefore $|pn[u, D]| \geq 2$ for all $u \in D$.

Case B: $G = K_n$, $n \geq 2$. Here $ir(G) = 1$ and every vertex constitutes an $ir$-set of $G$. Let $D$ be any $ir$-set of $G$. Then $D = \{ u \}$ for some $u \in V(G)$. $|pn[u, D]| = n \geq 2$. □

Remark 3.4. Let $G$ be the graph obtained from $K_{n,n}$ by removing a 1-factor. Then $G$ is just $ir$-excellent.

Proof. If $V_1$ and $V_2$ are the partite sets and if $V_1 = \{ u_1, u_2, \ldots, u_n \}$, $V_2 = \{ v_1, v_2, \ldots, v_n \}$ and $u_i$ and $v_i$ are not adjacent ($1 \leq i \leq n$), then the $ir$-sets are $\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_n, v_n\}$. □

Theorem 3.1. Let $G$ be a graph of order $n$. Then $G$ is $ir$-excellent if and only if the following conditions hold.

(i) $ir(G)$ divides $n$.

(ii) $G$ has exactly $\frac{n}{ir(G)}$ distinct $ir$-sets.

Proof. (i) Let $G$ be just $ir$-excellent. Then $G$ can be partitioned into $t$ sets each of which is an $ir$-set. Therefore $t \cdot ir(G) = n$. Therefore $ir(G)$ divides $n$. 

(ii): \( V(G) = S_1 \cup S_2 \cup \cdots \cup S_m \) where each \( S_i \) is an \( ir \)-set of \( G \) and these sets are pairwise disjoint. Therefore there are \( m \) distinct \( ir \)-sets of \( G \) where \( m = \frac{n}{\text{ir}(G)} \). Suppose there exists an \( ir \)-set \( T \) different from \( S_1, S_2, \cdots, S_m \). Since \( S_1 \cup S_2 \cup \cdots \cup S_m = V(G) \supseteq T \), every element \( x \in T \) belongs to some \( S_i \), \( 1 \leq i \leq m \). Therefore \( x \) belongs to two \( ir \)-sets of \( G \), a contradiction.

Conversely, suppose the three conditions hold. Let \( m = \frac{n}{\text{ir}(G)} \). By (iii) \( G \) has exactly \( m \) distinct \( ir \)-sets. Suppose \( V = S_1 \cup S_2 \cup \cdots \cup S_m \) is a decomposition of \( V(G) \) where each \( S_i \) is a maximal irredundant set, \( 1 \leq i \leq m \). Then \( n = \sum_{i=1}^{m} |S_i| \geq m \text{ir}(G) \). Therefore each \( S_i \) is an \( ir \)-set of \( G \). Since \( G \) has exactly \( m \) distinct \( ir \)-sets, \( S_1, S_2, \cdots, S_m \) are the distinct \( ir \)-sets of \( G \) and hence \( V = S_1 \cup S_2 \cup \cdots \cup S_m \) is a partition into disjoint \( ir \)-sets of \( G \). Therefore each vertex \( v \) belongs to exactly one \( S_i \), for some \( i \), \( 1 \leq i \leq m \). Therefore \( G \) is just \( ir \)-excellent.

**Theorem 3.2.** Every graph is an induced subgraph of a just \( ir \)-excellent graph.

**Proof.** Let \( G \) be a given graph. If \( G \) is just \( ir \)-excellent, then there is nothing to prove. Assume that \( G \) is not just \( ir \)-excellent. Let \( V(G) = \{v_1, v_2, \cdots, v_n\} \). Consider the cycle \( C_{3n} \). It is just \( ir \)-excellent. Let \( S_1, S_2, S_3 \) be the distinct \( ir \)-sets of \( C_{3n} \). Label the vertices of \( S_1 \) by \( u_1, u_2, u_3, \cdots, u_n \). Now in \( C_{3n} \) we add edges \( u_i u_j \) if and only if \( v_i v_j \) is an edge in \( G \). Let the resulting graph be \( H \). Then the induced subgraph \( \langle S_1 \rangle \) in \( H \) is isomorphic to \( G \). By theorem 4.12 in [6], \( H \) is just \( ir \)-excellent and \( \text{ir}(H) = n \). Every \( ir \)-set is a \( \gamma \) set. Thus the given graph \( G \) is an induced subgraph of a just \( ir \)-excellent graph \( H \).

**Example 3.2.**

\[ G \]

\[ H \]

\( G \) is an induced subgraph of \( H \) which is a just \( ir \)-excellent graph.

**References**


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