A STUDY ON
THE INHERENT INJ-EQUITABLE GRAPHS

Hanaa Alashwali, Ahmad N. Alkenani, A. Saleh, and Najat Muthana

ABSTRACT. Let $G$ be a graph. The inherent Inj-equitable graph of a graph $G$ ($IIE(G)$) is the graph with the same vertices as $G$ and any two vertices $u$ and $v$ are adjacent in $IIE(G)$ if they are adjacent in $G$ and $|\text{deg}_{in}(u) - \text{deg}_{in}(v)| \leq 1$, where for any vertex $w \in V(G)$, $\text{deg}_{in}(w) = |\{w' \in V : N(w') \cap N(u) \neq \phi\}|$. In this paper, inherent Inj-equitable graph of some graphs are obtained, some properties and results are established. We define iterated Inj-equitable graph of a graph, complete Inj-equitable graph and we define the Inj-equitable graph.

1. Introduction

All graphs considered in this paper are finite, undirected without loops or multiple edges. Let $G = (E, V)$ be a graph with vertex set $V = \{v_1, v_2, ..., v_n\}$. Thus $|V| = n$. The open neighborhood and the closed neighborhood of $v$ are denoted by $N(v) = \{u \in V(G) : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. The degree of a vertex $v$ in $G$ is $\deg(v) = |N(v)|$. $\Delta(G)$ and $\delta(G)$ are the maximum and minimum vertex degree of $G$ respectively. The distance $d(u, v)$ between any two vertices $u$ and $v$ in a graph $G$ is the number of the edges in a shortest path. The eccentricity of a vertex $u$ in a connected graph $G$ is $e(u) = \max\{d(u, v), v \in V\}$. The diameter of $G$ is the value of the greatest eccentricity, and the radius of $G$ is the value of the smallest eccentricity. The Inj-neighborhood of a vertex $u \in V(G)$ denoted by $N_{in}(u)$ is defined as $N_{in}(u) = \{v \in V(G) : |\Gamma(u, v)| \geq 1\}$, where $|\Gamma(u, v)|$ is the number of common neighborhood between the vertices $u$ and $v$. The cardinality of $N_{in}(u)$ is called injective degree of the vertex $u$ and is denoted by $\deg_{in}(u)$ in $G$ and $N_{in}[u] = N_{in}(u) \cup \{u\}$. Let $G$ and $H$ be any two graphs with vertex sets $V(G)$, $V(H)$ and edge sets $E(G)$, $E(H)$, respectively. Then the union $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The join $G \_\_ H$ is the graph obtained by taking the disjoint union of $G$ and $H$ and adding all edges

2010 Mathematics Subject Classification. 26D05, 26D07.

Key words and phrases. Inherent Inj-equitable graph, equitable matrix, injective matrix.
The corona product \( G \circ H \) is obtained by taking one copy of \( G \) and \(|V(G)|\) copies of \( H \) and by joining each vertex of the \( i \)-th copy of \( H \) to the \( i \)-th vertex of \( G \), where \( 1 \leq i \leq |V(G)| \). The cartesian product \( G \times H \) is a graph with vertex set \( V(G) \times V(H) \) and edge set \( E(G \times H) = \{(u, u'), (v, v') : u = v \text{ and } u, v \in E(G), \text{ or } u = v' \text{ and } (u, v) \in E(G)\} \). For more terminologies and notations, we refer the reader to [2], [4], [6] and [8]. A strongly regular graph with parameters \((n, k, \lambda, \mu)\) is a \( k \)-regular graph with \( n \) vertices such that any two adjacent vertices have \( \lambda \) common neighbors, and any two non-adjacent vertices have \( \mu \) common neighbors, [5].

**Definition 1.1** ([1]). Let \( G = (V, E) \) be a graph. The inherent injective equitable graph of \( G \), denoted by \( IIE(G) \), is defined as the graph with vertex set \( V(G) \) and two vertices \( u \) and \( v \) are adjacent in \( IIE(G) \) if and only if they are adjacent in \( G \) and \(|\text{deg}_{in}(u) - \text{deg}_{in}(v)| \leq 1\). An edge \( e = uv \in G \) is called injective equitable edge if \(|\text{deg}_{in}(u) - \text{deg}_{in}(v)| \leq 1\) and we say that \( u \) and \( v \) are Inj-equitable adjacent.

The adjacency matrix of the graph \( G \) is the symmetric square matrix \( A = A(G) = ||a_{ij}|| \) of order \( n \) whose \((i, j)\)-entry is defined as:

\[
(1.1) \quad a_{ij} = \begin{cases} 
1 & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent;} \\
0 & \text{otherwise.}
\end{cases}
\]

The equitable graph of a graph \( G \) is the graph with vertex set \( V(G) \) and two vertices \( u \) and \( v \) are adjacent if and only if \(|\text{deg}(u) - \text{deg}(v)| \leq 1\), [7]. The adjacency matrix of equitable graph is the symmetric square matrix \( A_e = A_e(G) = ||b_{ij}|| \) whose \((i, j)\)-entry is defined as:

\[
(1.2) \quad b_{ij} = \begin{cases} 
1 & \text{if } v_i \text{ and } v_j \text{ are adjacent and } |\text{deg}(v_i) - \text{deg}(v_j)| \leq 1; \\
0 & \text{otherwise.}
\end{cases}
\]

The adjacency matrix of the congraph, defined in [3], is the symmetric matrix \( ||a_{ij}'|| \) whose \((i, j)\)-entry is defined as:

\[
(1.3) \quad a_{ij}' = \begin{cases} 
1 & \text{if } |\Gamma(v_i, v_j)| \geq 1; \\
0 & \text{otherwise.}
\end{cases}
\]

Where \( \Gamma(v_i, v_j) \) is the set of vertices, different from \( v_i \) and \( v_j \), that are adjacent to both \( v_i \) and \( v_j \).

Bearing in mind equations 1.2 and 1.3 as a sort of compromise, we introduce a new symmetric square matrix \( A_{IIE} = ||d_{ij}|| \) of order \( n \), whose \((i, j)\)-entry is defined as:

\[
(1.4) \quad d_{ij} = \begin{cases} 
1 & \text{if } v_i \text{ and } v_j \text{ are adjacent and } |\text{deg}_{in}(v_i) - \text{deg}_{in}(v_j)| \leq 1; \\
0 & \text{otherwise.}
\end{cases}
\]

This matrix can be viewed as the adjacency matrix of the inherent injective equitable graph. In this paper, the benefit of graph characterization to study the properties and the structure of graphs motivated us to introduce and study new
graphs called inherent injective equitable graph and complete inherent injective equitable graph.

2. The Inherent Inj-equitable Graph of a Graph

In this section, we discuss some properties of the inherent injective equitable graph of a graph and the inherent injective equitable graph of some graph’s families is found.

**Proposition 2.1.** For any graph $G$, $G \cong IIE(G)$ if and only if every edge is an Inj-equitable edge.

**Theorem 2.1.** Let $G$ be a complete graph or $k$-regular triangle-free graph with diameter 2, then $IIE(G) \cong G$.

**Proof.** If $G$ is complete graph, then obviously $IIE(G) \cong G$. Suppose that $G$ is $k$-regular triangle-free graph with diameter 2. We know that $IIE(G)$ is a subgraph of $G$ for any graph $G$. Since $G$ is $k$-regular with diameter 2, then for any vertex $v$, $\deg(v) = k$ and $\deg_{in}(v) = n - k - 1$. So, any adjacent vertices in $G$ is also Inj-equitable adjacent. Hence, $IIE(G) \cong G$.

**Corollary 2.1.** For any strongly regular graph without triangle $G$, $IIE(G) \cong G$.

**Proposition 2.2.** For any strongly regular graph with parameters $(n, k, \lambda, \eta)$, $IIE(G)$ is also a strongly regular graph with the same parameters.

**Proof.** Let $G$ be a strongly regular graph with parameters $(n, k, \lambda, \eta)$. Then for any two adjacent vertices $u$ and $v$, $\deg_{in}(u) = \deg_{in}(v) = \lambda$. Therefore, $|\deg_{in}(u) - \deg_{in}(v)| = 0$.

Hence $IIE(G) \cong G$.

**Remark 2.1.** It is not true in general that for any regular graph $G$, $IIE(G) \cong G$. For example, one can see Figure 1.

**Figure 1.** A regular graph $G$ with $IIE(G) \cong G$

**Proposition 2.3.** The following holds:

(i) For any path $P_n$, $IIE(P_n) \cong P_n$. 

(ii) For any cycle \( C_n \), \( IIE(C_n) \cong C_n \).
(iii) For any wheel \( W_n \), \( IIE(W_n) \cong W_n \).

**Proposition 2.4.** For any complete bipartite graph \( K_{r,s} \), where \( r + s \geq 4 \),

\[
IIE(K_{r,s}) \cong \begin{cases} 
K_{r,s} & \text{if } |r-s| \leq 1; \\
K_{r+s} & \text{if } |r-s| \geq 2.
\end{cases}
\]

**Proof.** Let \( G \cong K_{r,s} \) be a complete bipartite graph with partite sets \( A \) and \( B \) such that \( |A| = r, |B| = s \). Clearly for any vertex \( v \) from \( A \), \( \deg_{in}(v) = r-1 \) and for any vertex \( u \) from \( B \), \( \deg_{in}(u) = s-1 \). Therefore, \( u \) and \( v \) are Inj-equitable adjacent if \( |(s-1) - (r-1)| = |r-s| \leq 1 \). Otherwise, they are not Inj-equitable adjacent. Hence,

\[
IIE(K_{r,s}) \cong \begin{cases} 
K_{r,s} & \text{if } |r-s| \leq 1; \\
K_{r+s} & \text{if } |r-s| \geq 2.
\end{cases}
\]

\[\blacksquare\]

A firefly graph \( F_{r,s,t} \) is a graph on \( 2r + 2s + t + 1 \) vertices that consists of \( r \) triangles, \( s \) pendant paths of length 2 and \( t \) pendant edges sharing a common vertex.

![Figure 2. Firefly Graph](image)

**Theorem 2.2.** For any firefly graph \( F_{r,s,t} \), where \( r, s, t \geq 1 \),

\[
IIE(F_{r,s,t}) \cong \begin{cases} 
F_{r,0,s+1} \cup K_s & \text{if } t = 1; \\
rK_2 \cup K_{1,s+2} \cup K_s & \text{if } t = 2; \\
K_{2s+t+1} \cup rK_2 & \text{if } t > 2.
\end{cases}
\]
Let $K$ be any firefly graph $F_{r,s,t}$ as in Figure 2., where $r \geq 1, s \geq 1$ and $t \geq 1$. Let $v$ be the center vertex, $v_i$ where $i = 1, 2, \ldots, 2r$ be any vertex from the triangle other than $v$, $w_i$ where $i = 1, 2, \ldots, t$ be any end vertex in the pendant edge, $u_i$ and $u_i'$ where $i = 1, 2, \ldots, s$ be any end vertex and internal vertex respectively in the pendant path. Then, $\deg_{in}(v) = 2r + s$, $\deg_{in}(v_i) = 2r + s + t$, $\deg_{in}(w_i) = 2r + s + t - 1$, $\deg_{in}(u_i) = 2r + s + t - 1$ and $\deg_{in}(u_i') = 1$. We have three cases:

**Case 1.** Suppose that, $t = 1$. Since $|\deg_{in}(v) - \deg_{in}(v_i)| = 1$ and

$$|\deg_{in}(u_i') - \deg_{in}(w_i)| = 0,$$

then $IIE(F_{r,s,t}) \cong F_{r,0,s+1} \cup K_s$.

**Case 2.** Suppose that, $t = 2$. Then, $\deg_{in}(v) = 2r + s$, $\deg_{in}(v_i) = 2r + s + 2$, $\deg_{in}(w_i) = 2r + s + 1$, $\deg_{in}(u_i') = 2r + s + 1$. Hence, $IIE(F_{r,s,t}) \cong rK_2 \cup K_1 \cup K_{s+2} \cup K_s$.

**Case 3.** Suppose that $t > 2$, then the only injective edges are $e_1 = v_1v_2$, $e_2 = v_2v_3 \ldots e_r = v_2v_1v_2r$. Hence, $IIE(F_{r,s,t}) \cong K_{2s+t+1} \cup rK_2$. $\square$

**Proposition 2.5.**

(i) For any firefly graph $G \cong F_{r,0,0}$, $IIE(G) \cong G$.

(ii) For any firefly graph $F_{0,s,0}$,

$$IIE(F_{0,s,0}) \cong \begin{cases} F_{0,2,0} & \text{if } s = 2; \\ K_{1,s} \cup K_s & \text{if } s > 2. \end{cases}$$

(iii) For any firefly graph $F_{r,s,t}$, where $r = s = 0$

$$IIE(F_{r,s,t}) \cong \begin{cases} F_{0,0,t} & \text{if } t \leq 2; \\ K_{t+1} & \text{if } t \geq 3. \end{cases}$$

(iv) For any firefly graph $F_{r,s,t}$, where $r = 0, s \geq 1, t \geq 1$,

$$IIE(F_{r,s,t}) \cong \begin{cases} P_4 & \text{if } s = t = 1; \\ K_{t+s} \cup K_s & \text{if } t \leq 2; \\ K_{t+2s+1} & \text{if } t \geq 3. \end{cases}$$

**Proposition 2.6.** For any bipartite graph $G$, $IIE(G)$ is also bipartite graph.

**Proof.** Let $G$ be a bipartite graph. Suppose that $IIE(G)$ is not bipartite graph. Then it contains at least one odd cycle say $C_m$. Since $IIE(G)$ is a subgraph of $G$, then $G$ contains odd cycle which contradicts that $G$ is bipartite graph. Hence $IIE(G)$ is bipartite graph. $\square$

**Proposition 2.7.** Let $G$ be a graph such that $G \cong P_m \times P_2$. Then $IIE(G) \cong P_m \times P_2$. 

Figure 3. $P_m \times P_2$.

Proof. Let $G$ be a graph such that $G \cong P_m \times P_2$. Then we have four cases:

Case 1. If $m = 2$, then $G \cong C_4$. Therefore, $IIE(G) \cong G$.

Case 2. If $m = 3$, then $\deg_{\text{in}}(v) = 2$ for all $v \in V(G)$. Therefore, for any adjacent vertices $u$ and $v$, $|\deg_{\text{in}}(u) - \deg_{\text{in}}(v)| = 0$. Hence, $IIE(G) \cong P_m \times P_2$.

Case 3. If $m = 4$, then for all $v \in V(G)$, either $\deg_{\text{in}}(v) = 2$ or $\deg_{\text{in}}(v) = 3$. Therefore, for any two adjacent vertices $u$ and $v$, $|\deg_{\text{in}}(u) - \deg_{\text{in}}(v)| \leq 1$. Hence, $IIE(G) \cong P_m \times P_2$.

Case 4. If $m \geq 5$, let $G$ be labeling as in Figure 3. Then, $\deg_{\text{in}}(v_1) = \deg_{\text{in}}(v_m) = \deg_{\text{in}}(u_1) = \deg_{\text{in}}(u_m) = 2$, $\deg_{\text{in}}(v_2) = \deg_{\text{in}}(v_{m-1}) = \deg_{\text{in}}(u_2) = \deg_{\text{in}}(u_{m-1}) = 3$ and for $i = 3, 4, ... m - 2$, $\deg_{\text{in}}(v_i) = \deg_{\text{in}}(u_i) = 4$. Therefore, for any two adjacent vertices $u$ and $v$ in $G$, $|\deg_{\text{in}}(u) - \deg_{\text{in}}(v)| \leq 1$. Hence, $IIE(G) \cong P_m \times P_2$.

\[\square\]

Proposition 2.8. Let $G$ be a graph such that $G \cong P_m \times P_3$, where $m \geq 4$. Then $IIE(G) \cong P_m \times P_3 - \{e_1, e_2\}$, where $e_1$ and $e_2$ are the edges which are not Inj-equitable edges in $G$.

Figure 4. $P_m \times P_3$

Proof. Suppose that, $G \cong P_m \times P_3$ be labeling as in Figure 4. Then we have two cases:

Case 1. If $m = 4$, all the vertices have Inj-equitable degree 4 except $v_{22}$ and $v_{23}$ have Inj-degree 5. Therefore all the edge are In-equitable edge except $e_1 = v_{21}v_{22}$ and $e_2 = v_{23}v_{24}$. Hence, $IIE(G) \cong P_m \times P_3 - \{e_1, e_2\}$. 


Case 2. If $m \geq 5$, then all the edges are Inj-equitable edges except $e_1 = v_{21}v_{22}$ and $e_2 = v_{2m-1}v_{2m}$. Hence, $\text{IIE}(G) \cong P_m \times P_3 - \{e_1, e_2\}$.

For the generalized case, we have the following result:

**Proposition 2.9.** Let $G$ be a graph such that $G \cong P_m \times P_n$, where $m, n \geq 5$. Then $\text{IIE}(G) \cong C_{2m+2n-4} \cup (P_{m-2} \times P_{n-2})$.

![Figure 5. $P_m \times P_n$](image)

**Proof.** Suppose that $G \cong P_m \times P_n$ be labeling as in Figure 5. Then
\[
\deg_{\text{in}}(v_{11}) = \deg_{\text{in}}(v_{1m}) = \deg_{\text{in}}(v_{n1}) = \deg_{\text{in}}(v_{nm}) = 3
\]
and
\[
\deg_{\text{in}}(v_{12}) = \deg_{\text{in}}(v_{(1)(m-1)}) = \deg_{\text{in}}(v_{21}) = \deg_{\text{in}}(v_{2m}) = \deg_{\text{in}}(v_{(n-1)(1)}) = \deg_{\text{in}}(v_{(n-1)(m)}) = \deg_{\text{in}}(v_{n2}) = \deg_{\text{in}}(v_{(n)(m-2)}) = 4.
\]

Also, for $i = 3, 4, \ldots, m - 2$,
\[
\deg_{\text{in}}(v_{1i}) = \deg_{\text{in}}(v_{ni}) = 5
\]
and for $i = 3, 4, \ldots, m$,
\[
\deg_{\text{in}}(v_{i1}) = \deg_{\text{in}}(v_{im}) = 5.
\]

For $i, j = 2, m - 1$,
\[
\deg_{\text{in}}(v_{ij}) = 6.
\]

For $i = 3, 4, \ldots, m - 2$,
\[
\deg_{\text{in}}(v_{2i}) = \deg_{\text{in}}(v_{(n-1)(i)}) = 7
\]
and for $i = 3, 4, \ldots, n - 1$,
\[
\deg_{\text{in}}(v_{12}) = \deg_{\text{in}}(v_{i(m-1)}) = 7.
\]
For $i, j = 3, 4, \ldots, m - 2$,
\[ \deg_{in}(v_{ij}) = 8. \]
Therefore, all the edges are Inj-equitable edges except
\[ v_{21}v_{22}, v_{(2)(m-1)}v_{2m}, v_{(n-1)(1)}v_{(n-1)(2)}, v_{(n-1)(m-1)}v_{(n-1)(m)} \]
and for $j = 2, 3, \ldots, m - 1$, $v_{ij}v_{2j}, v_{nj}v_{(n-1)(j)}$. Hence $IIE(P_m \times P_n) \cong C_{2m+2n-4} \cup (P_{m-2} \times P_{n-2})$. 

**Proposition 2.10.** Let $G$ be a generalized Petersen graph $GP(m, 1)$. Then $IIE(G) \cong GP(m, 1)$.

**Proof.** Let $G$ be a generalized Petersen graph $GP(m, 1)$. Then $G \cong C_m \times P_2$. We have three cases:

**Case 1.** If $m = 3$, then for all $v \in V(G)$, $\deg_{in}(v) = 4$. Therefore, all the edges are Inj-equitable edges. Hence, $IIE(G) \cong C_m \times P_2$.

**Case 2.** If $m = 4$, then for all $v \in V(G)$, $\deg_{in}(v) = 3$. Therefore, all the edges are Inj-equitable edges. Hence, $IIE(G) \cong C_m \times P_2$.

**Case 3.** If $m \geq 5$, let $G \cong C_m \times P_2$ be labeling as in Figure 6. Then $\deg_{in}(v_i) = 4$ and $\deg_{in}(u_i) = 4$, for $i = 1, 2, \ldots, m$. Therefore, for any adjacent vertices $u$ and $v$ in $G$, $|\deg_{in}(u) - \deg_{in}(v)| = 0$. Therefore, $IIE(G) \cong C_m \times P_2$. Hence, $IIE(GP(m, 1)) \cong GP(m, 1)$. 

**Proposition 2.11.** Let $G \cong C_m \times P_3$. Then $IIE(G) \cong C_m \times P_3$.

**Proof.** Let $G \cong C_m \times P_3$ be labeling as in Figure 7. We have three cases:

**Case 1.** If $m = 3$, then for all $v \in V(G)$, $\deg_{in}(v) = 5$. Therefore, all the edges are Inj-equitable edges. Hence, $IIE(G) \cong C_m \times P_3$. 

![Figure 6. GP(m,1)](image-url)
Case 2. If $m = 4$, then for $i = 1, 2, \ldots, 4$, $\deg_{in}(v_{1i}) = \deg_{in}(v_{3i}) = 4$ and $\deg_{in}(v_{2i}) = 5$. Therefore, all the edges are Inj-equitable edges. Hence, $\text{IIE}(G) \cong C_m \times P_3$.

Case 3. If $m \geq 5$, then for $i = 1, 2, \ldots, m$, $\deg_{in}(v_{1i}) = \deg_{in}(v_{3i}) = 5$ and $\deg_{in}(v_{2i}) = 6$. Therefore, for any adjacent vertices $u$ and $v$ in $G$, $|\deg_{in}(u) - \deg_{in}(v)| = 0$. Hence, $\text{IIE}(G) \cong C_m \times P_3$. □

Theorem 2.3. For any graph $G$ such that $G \cong C_m \times P_n$, $\text{IIE}(G) \cong 2C_m \cup (C_m \times P_{n-2})$, where $n \geq 5$. 

![Figure 7. $C_m \times P_3$](image1)

![Figure 8. $C_m \times P_n$](image2)
Suppose that \( G \cong C_m \times P_n \) be labeling as in Figure 8. We have three cases:

**Case 1.** If \( m = 3 \), then for \( i = 1, 2, 3 \), \( \deg_{in}(v_{1i}) = \deg_{in}(v_{ni}) = 5 \), \( \deg_{in}(v_{2i}) = \deg_{in}(v_{n-1 i}) = 7 \) and for \( i = 3, 4, \ldots, n - 2 \), \( j = 1, 2, 3 \), \( \deg_{in}(v_{ij}) = 8 \). Hence \( IIE(G) \cong 2C_m \cup (C_m \times P_{n-2}) \).

**Case 2.** If \( m = 4 \), then for \( i = 1, 2, \ldots, 4 \), \( \deg_{in}(v_{1i}) = \deg_{in}(v_{ni}) = 4 \), \( \deg_{in}(v_{2i}) = \deg_{in}(v_{n-1 i}) = 6 \) and for \( i = 3, 4, \ldots, n - 2 \), \( j = 1, 2, \ldots, 4 \), \( \deg_{in}(v_{ij}) = 7 \). \( IIE(G) \cong 2C_m \cup (C_m \times P_{n-2}) \).

**Case 3.** If \( m \geq 5 \), then as in Figure 8., for \( i = 1, 2, \ldots, m \), \( \deg_{in}(v_{1i}) = \deg_{in}(v_{mi}) = 5 \), \( \deg_{in}(v_{2i}) = \deg_{in}(v_{m-1 i}) = 7 \) and for \( i = 3, 4, \ldots, n - 2 \), \( j = 1, 2, \ldots, m \), \( \deg_{in}(v_{ij}) = 8 \). \( IIE(G) \cong 2C_m \cup (C_m \times P_{n-2}) \).

**Theorem 2.4.** For any graph \( G \), such that \( G \cong C_n \times C_m \), \( IIE(G) \cong C_n \times C_m \).

**Figure 9.** \( C_n \times C_m \)

Proof. Let \( G \) be any graph such that \( G \cong C_n \times C_m \). From Figure 9., for all \( v \in V(G) \), \( \deg_{in}(v) = 8 \). Therefore, all the edges are Inj-equitable edges. Hence \( IIE(G) \cong C_n \times C_m \).

**Proposition 2.12.** For any two graphs \( G_1 \) and \( G_2 \), \( IIE(G_1 \vee G_2) = G_1 \vee G_2 \).

Proof. Let \( G_1 \) and \( G_2 \) be any two graphs. Since every edge in \( G_1 \vee G_2 \) is injective equitable edge, then \( IIE(G_1 \vee G_2) = G_1 \vee G_2 \).

**Proposition 2.13.** For any cycle graph \( C_n \) and any totally disconnected graph \( K_m \), where \( m > 2 \), \( IIE(C_n \circ K_m) \cong C_n \cup K_{nm} \).

Proof. Let \( \{u_1, u_2, \ldots, u_n\} \) be the vertex set of the cycle graph \( C_n \), and let \( \{v_1, v_2, \ldots, v_m\} \) be the vertex set of \( K_m \). Then for \( i = 1, 2, \ldots, n \), \( \deg_{in}(u_i) = 2(m+1) \) and for \( j = 1, 2, \ldots, m \), \( \deg_{in}(v_j) = m + 1 \). Therefore, \( |\deg_{in}(u_i) - \deg_{in}(v_j)| = m + 1 > 1 \). Hence, \( IIE(C_n \circ K_m) \cong C_n \cup K_{nm} \).
Theorem 2.5. For any graph $G$ with $\delta \geq 2$, if $G$ is $k$-regular or $(k, k + 1)$-biregular, then

$$IIE(G \circ \overline{K_m}) = IIE(G) \cup \overline{K_{nm}}.$$ 

where $n$ is the number of vertices in $G$.

Proof. Let $G$ be a $k$-regular graph with $n$ vertices and $\delta \geq 2$. Suppose that $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_m\}$ is the vertex set of $G$ and $\overline{K_m}$, respectively. Therefore, Then for $i = 1, 2, \ldots, n$, $\deg_{in}(u_i) = k(m + 1)$ and for $j = 1, 2, \ldots, m$, $\deg_{in}(v_j) = k + m - 1$. Therefore, $|\deg_{in}(u_i) - \deg_{in}(v_j)| = m(k - 1) + 1 > 1$. Hence, $IIE(G \circ \overline{K_m}) = IIE(G) \cup \overline{K_{nm}}$. Similarly, we can prove if $G$ is $(k, k + 1)$-biregular, then $IIE(G \circ \overline{K_m}) = IIE(G) \cup \overline{K_{nm}}$. \hfill $\square$

3. Complete inherent Inj-equitable graphs

Definition 3.1. A graph $G$ is called complete inherent injective equitable graph (CIIE-graph) if for any two adjacent vertices $u$ and $v$, $|\deg_{in}(u) - \deg_{in}(v)| \leq 1$.

Example 3.1. $C_n \times C_m$ is CIIE-graph.

Proposition 3.1. Any complete graph is CIIE-graph but the converse is not always true. For example, paths and cycles are CIIE-graph but not complete.

Proposition 3.2. Let $G$ be any graph. $IIE(G) \cong G$ if and only if $G$ is CIIE-graph.

Proposition 3.3. Let $H$ be a CIIE-graph and let $G$ be a subgraph of $H$. Then $IIE(G)$ is a subgraph of $IIE(H)$.

Proof. Let $H$ be a CIIE-graph and let $G$ be a subgraph of $H$. Let $e$ be an edge in $IIE(G)$. Then $e \in G$. Therefore $e \in H$. So, $e \in IIE(H)$. Hence, $IIE(G)$ is a subgraph of $IIE(H)$. \hfill $\square$

Proposition 3.4. For any CIIE-graph $G$, $IIE(G)$ is also CIIE-graph.

Proof. Let $e = uv$ be any edge in $IIE(G)$. Then $e$ is an edge in $G$. Since $G$ is CIIE-graph, then $|\deg_{in}(u) - \deg_{in}(v)| \leq 1$ in $G$. Therefore, $|\deg_{in}(u) - \deg_{in}(v)| \leq 1$ in $IIE(G)$. So, $IIE(G)$ is CIIE-graph. \hfill $\square$

Theorem 3.1. A graph $G$ is CIIE-graph if and only if $A(G) = A_{IIE}(G)$, where $A(G)$ and $A_{IIE}(G)$ are the adjacency matrix of $G$ and adjacency matrix of the inherent injective equitable graph of $G$ respectively.

Proof. Suppose that $G$ is CIIE-graph. Then for any two adjacent vertices $v_i$ and $v_j$, $|\deg_{in}(v_i) - \deg_{in}(v_j)| \leq 1$. Therefore, $A(G) = A_{IIE}(G)$. Similarly, if $A(G) = A_{IIE}(G)$ then $G$ is CIIE-graph. \hfill $\square$

Proposition 3.5. Let $G \cong \bigcup_{i=1}^{m} G_i$. If $G_i, i = 1, 2, \ldots, m$, are CIIE-graphs, then $G$ is CIIE-graph.
Proof. Let u and v be any two adjacent vertices in G. Therefore, u and v are adjacent vertices in a graph $G_i$, $i = 1, 2, \ldots, n$. But $G_i$ is C1IE-graph. Then, $|\text{deg}_{in}(u) - \text{deg}_{in}(v)| \leq 1$. Hence, G is C1IE-graph.

Definition 3.2. A graph G which is C1IE-graph is called strong C1IE-graph if $\overline{G}$ is also C1IE-graph.

Example 3.2. Any cycle $C_n$ with n vertices is strong C1IE-graphs. Similarly, any path $P_n$ with n vertices is strong C1IE-graphs.

Proposition 3.6. For any graph G $\cong K_{m,n}$ such that $|m - n| \leq 1$, G is strong C1IE-graph.

Proof. Let u and v be any two adjacent vertices in G $\cong K_{m,n}$. Then $|\text{deg}_{in}(u) - \text{deg}_{in}(v)| \leq 1$. Therefore, G is C1IE-graph. Also, since $\overline{G} \cong K_m \cup K_n$, then $|\text{deg}_{in}(u) - \text{deg}_{in}(v)| \leq 1$ for any two adjacent vertices u and v. Hence, G is strong C1IE-graph.

Proposition 3.7. For any graph G, IIE($\overline{G}$) is a subgraph of $\overline{\text{IIE}(G)}$.

Proof. Let $e$ be any edge in IIE($\overline{G}$). Then, $e \in \overline{G}$. Therefore, $e \notin G$. So, $e \notin \text{IIE}(G)$. Then, $e \in \overline{\text{IIE}(G)}$. Hence $\text{IIE}(G)$ is a subgraph of $\text{IIE}(G)$.

Proposition 3.8. $\overline{\text{IIE}(G)}$ is subgraph of $\overline{G}$.

Theorem 3.2. For any strong C1IE-graph G, $\overline{\text{IIE}(G)} = \text{IIE}(\overline{G})$.

Proof. Let $e$ be any edge in $\overline{\text{IIE}(G)}$. Then $e \notin \text{IIE}(G)$. Since G is strong C1IE-graph, then $e \notin G$. Therefore, $e \in \overline{G}$ which implies that $e \in \text{IIE}(\overline{G})$, since G is strong C1IE-graph. So, $\overline{\text{IIE}(G)} \subseteq \text{IIE}(\overline{G})$. Hence by proposition 3.7, $\overline{\text{IIE}(G)} = \text{IIE}(\overline{G})$.

Theorem 3.3. Let G be a graph with adjacency matrix $A = ||a_{ij}||$. Let $B_{\text{IIE}} = \|b_{ij}\|$, where $b_{ij} = \begin{cases} 1 & \text{if } |\text{deg}_{in}(v_i) - \text{deg}_{in}(v_j)| \leq 1; \\ 0 & \text{otherwise.} \end{cases}$

Then

$$A_{\text{IIE}} = \|b_{ij}\| = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & \cdots & a_{1n}b_{1n} \\ \vdots & \ddots & \cdots & \vdots \\ a_{n1}b_{n1} & a_{n2}b_{n2} & \cdots & a_{nn}b_{nn} \end{bmatrix},$$

where $A_{\text{IIE}}$ is the adjacency matrix of the inherent injective equitable graph of G.

Proof. Suppose that G is a graph with adjacency matrix $A$ and suppose $B_{\text{IIE}} = \|b_{ij}\|$, where $b_{ij} = \begin{cases} 1 & \text{if } |\text{deg}_{in}(v_i) - \text{deg}_{in}(v_j)| \leq 1; \\ 0 & \text{otherwise.} \end{cases}$

Let
We consider iterated inherent Inj-equitable graph, i.e., those

If

Then for \(i, j = 1, 2, \ldots, m\), \(a_{ij}b_{ij} = 0\) if \(a_{ij} = 0\) or \(b_{ij} = 0\), i.e., \(v_i\) and \(v_j\) are not

For any graph \(G\) the inherent Inj-equitable graph is not unique. A graph \(G\) for some \(k\) equitable family of

Continues in the same way until we get \(C_{IIE}^{(k)}(G)\) such that

\[
C_{IIE} = \begin{bmatrix}
a_{11}b_{11} & a_{12}b_{12} & \cdots & a_{1n}b_{1n} \\
\vdots & & & \vdots \\
a_{n1}b_{n1} & a_{n2}b_{n2} & \cdots & a_{nn}b_{nn}
\end{bmatrix}.
\]

Therefore, \(C_{IIE}^{(k)}(G)\) are Inj-equitable adjacent. Therefore,

\[
C_{IIE} = \begin{cases}
1 & \text{if } v_i \text{ and } v_j \text{ are Inj-equitable adjacent; } \\
0 & \text{otherwise.}
\end{cases}
\]

\[
\text{Hence } C_{IIE} = A_{IIE}. \quad \square
\]

4. Iterated inherent Inj-equitable graphs

**Definition 4.1.** We consider iterated inherent Inj-equitable graph, i.e., those obtained from a graph \(G\) as follows: \(IIE^{0}(G) = G\) and \(IIE^{k} = IIE(IIE^{k-1}(G))\), for \(k \in \mathbb{N}\).

**Theorem 4.1.** For any graph \(G\), there exists a positive integer \(k\) such that \(IIE^{k}(G)\) is \(CIE\)-graph for some \(k\).

**Proof.** If \(G\) is \(CIE\)-graph, then \(IIE(G) \cong G\) and then, \(IIE(G)\) is \(CIE\)-graph. If \(G\) is not \(CIE\)-graph, then there exists an edge \(e = uv\) such that \(|\text{deg}_{in}(u) - \text{deg}_{in}(v)| > 1\). Therefore \(e \notin IIE(G)\) and all the edges in \(IIE(G)\) are Inj-equitable edge in \(G\). If \(IIE(G)\) is \(CIE\)-graph, then \(IIE^{2}(G)\) is \(CIE\)-graph. If it is not \(CIE\)-graph, then there exist an edge \(e\) in \(IIE(G)\) such that \(e\) is not Inj-equitable edge and therefore, \(e \notin IIE^{k}(G)\) and all the edges in \(IIE^{k}(G)\) are Inj-equitable edge in \(IIE(G)\). Continues in the same way until we get \(CIE\)-graph or totally disconnected graph. Hence there exists \(k \geq 1\) such that \(IIE^{k}(G)\) is \(CIE\)-graph. \(\square\)

**Definition 4.2.** For any graph \(G\), the completeness injective inherent equitable number is the smallest positive integer \(k\) such that \(IIE^{k}(G)\) is \(CIE\)-graph and denoted by \(c_{iie}(G)\).

**Proposition 4.1.**

(i) If \(G\) is \(CIE\)-graph, then \(c_{iie}(G) = 0\).

(ii) If \(G \cong C_{m} \times C_{n}\), then \(c_{iie}(G) = 0\).

5. Inherent Inj-equitable graphs

**Definition 5.1.** A graph \(G\) is said to be inherent Inj-equitable graph (IIE-graph) if there exists a graph \(H\) such that \(IIE(H) \cong G\).

For example, any path, cycle and complete graph are IIE-graph. The family of graphs \(H\) which satisfy the condition \(IIE(H) \cong G\) is called the inherent Inj-equitable family of \(G\) and denoted by

\[
G_{IIE} = \{H : IIE(H) \cong G\}.
\]

**Remark 5.1.** The inherent Inj-equitable graph is not unique.
Theorem 5.1. For any Complete bipartite graph $G \cong K_{1,p}$, $G$ is not IIE-graph, where $p \geq 3$.

Proof. Suppose to the contrary that $G \cong K_{1,p}$ is IIE-graph. So, there exists at least a graph $H$ such that $IIE(H) \cong G$. Therefore, $H$ contains at least the same number of edges as $G$ or more. Clearly $H \not\cong K_{1,p}$ and the number of edges in $H$ will be more than the number of edges in $K_{1,p}$. So any edge in $H$ other than the edges of $K_{1,p}$ is Inj-equitable edge which is contradiction to $IIE(H) \cong G$. Hence $G$ is not IIE-graph. \qed

References