PERMITING TRI-$f$-DERIVATIONS ON ALMOST DISTRIBUTIVE LATTICES

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Abstract. In this paper, we introduce the concept of permuting tri-$f$-derivation in an Almost Distributive Lattice (ADL) and derive some important properties of permuting tri-$f$-derivation in ADLs.

1. Introduction

The notion of derivation in lattices was first given in G. Szasz [14] in 1974. Several authors worked on derivations in Lattices ([1], [2], [3], [4], [5], [6], [15], [16] and [17]). The concept of derivation in an ADL was introduced in our earlier paper [8]. Further, in an ADL we worked on $f$-derivations in [9], symmetric bi-$f$-derivations in [10], symmetric bi-$f$-derivations in [11] and permuting tri-derivations in [12]. The concept of permuting tri-$f$-derivations in lattices was introduced by H. Yazarli and M. A. Öztürk [17] in 2011.

In this paper, we introduce the concept of permuting tri-$f$-derivations in an ADL and investigate some important properties. If $m$ is a maximal element in an ADL $L$, then we prove that $D(x, y, z) = fx$ when $fx \leq D(m, y, z)$ and if $fm$ is also a maximal element of $L$, then we prove that $D(x, y, z) = D(m, y, z)$ when $fx \geq D(m, y, z)$. Also, we prove that $fx \land D(x \lor y, z) = D(x, y, z)$ when $D$ is an isotone map and $fx \land D(x \lor y, z) \leq D(x, y, z)$ when $f$ is either a join preserving or an increasing function on $L$. We establish a set of conditions which are sufficient for a permuting tri-$f$-derivation on an ADL with a maximal element to become an isotone when $f$ is a homomorphism. Also, we prove

\[ d(x \land y) = (fy \land dx) \lor D(x, x, y) \lor D(x, y, y) \lor (fx \land dy) \]

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where \( d \) is the trace of a permuting tri-\( f \)-derivation on an associative \( \text{ADL} \) \( L \).

Finally, we prove that the set \( F_d(L) = \{ x \in L \mid dx = fx \} \) is a weak ideal in an associative \( \text{ADL} \) \( L \) where \( f \) is a join preserving map on \( L \).

2. Preliminaries

In this section, we recollect certain basic concepts and important results on Almost Distributive Lattices.

Definition 2.1. [7] An algebra \( (L, \lor, \land) \) of type \( (2, 2) \) is called an Almost Distributive Lattice, if it satisfies the following axioms:

\[ L_1 : (a \lor b) \land c = (a \land c) \lor (b \land c) \quad \text{(RD\land)} \]
\[ L_2 : a \land (b \lor c) = (a \land b) \lor (a \land c) \quad \text{(LD\land)} \]
\[ L_3 : (a \lor b) \land b = b \]
\[ L_4 : (a \lor b) \land a = a \]
\[ L_5 : a \lor (a \land b) = a \quad \text{for all } a, b, c \in L. \]

Definition 2.2. [7] Let \( X \) be any non-empty set. Define, for any \( x, y \in L, x \lor y = x \) and \( x \land y = y \). Then \( (X, \lor, \land) \) is an \( \text{ADL} \) and such an \( \text{ADL} \), we call discrete \( \text{ADL} \).

Throughout this paper \( L \) stands for an \( \text{ADL} \) \( (L, \lor, \land) \) unless otherwise specified.

Lemma 2.1. [7] For any \( a, b \in L \), we have:

\[ (i) \quad a \land a = a \]
\[ (ii) \quad a \lor a = a. \]
\[ (iii) \quad (a \land b) \lor b = b \]
\[ (iv) \quad a \land (a \lor b) = a \]
\[ (v) \quad a \lor (b \land a) = a. \]
\[ (vi) \quad a \lor b = a \quad \text{if and only if } a \land b = b \]
\[ (vii) \quad a \lor b = b \quad \text{if and only if } a \land b = a. \]

Definition 2.3. [7] For any \( a, b \in L \), we say that \( a \) is less than or equal to \( b \) and write \( a \leq b \), if \( a \land b = a \) or, equivalently, \( a \lor b = b \).

Theorem 2.1. [7] For any \( a, b, c \in L \), we have the following

\[ (i) \quad \text{The relation } \leq \text{ is a partial ordering on } L. \]
\[ (ii) \quad a \lor (b \land c) = (a \lor b) \land (a \lor c). \quad \text{(LD\lor)} \]
\[ (iii) \quad (a \lor b) \land a = a \lor (a \land b) \]
\[ (iv) \quad (a \lor b) \land c = (b \lor a) \land c. \]
\[ (v) \quad \text{The operation } \land \text{ is associative in } L. \]
\[ (vi) \quad a \land b \land c = b \land a \land c. \]

Theorem 2.2. [7] For any \( a, b \in L \), the following are equivalent.

\[ (i) \quad (a \land b) \lor a = a \]
\[ (ii) \quad a \land (b \lor a) = a \]
\[ (iii) \quad (b \land a) \lor b = b \]
\[ (iv) \quad b \land (a \lor b) = b \]
if it satisfies the following:

\( m \) (3) \( m \) (1) \( m \) is maximal

then

\( (x_1, x_2) \) with zero.

de\ defined a for all \( x \in L \) and equals to \( a \wedge b \).

\( \text{Definition 2.4. } [7] \) \( L \) is said to be associative, if the operation \( \vee \) in \( L \) is associative.

\( \text{Theorem 2.3. } [7] \) The following are equivalent:

\( (i) \) \( L \) is a distributive lattice.

\( (ii) \) The poset \( (L, \leq) \) is directed above.

\( (iii) \ a \wedge (b \vee a) = a \), for all \( a, b \in L \).

\( (iv) \) The operation \( \vee \) is commutative in \( L \).

\( (v) \) The operation \( \wedge \) is commutative in \( L \).

\( (vi) \) The relation \( \theta := \{(a, b) \in L \times L \mid a \wedge b = b\} \) is anti-symmetric.

\( (vii) \) The relation \( \theta \) defined in (vi) is a partial order on \( L \).

\( \text{Lemma 2.2. } [7] \) For any \( a, b, c, d \in L \), we have the following:

\( (i) \ a \wedge b \leq b \) and \( a \leq a \vee b \)

\( (ii) \ a \wedge b = b \wedge a \) whenever \( a \leq b \).

\( (iii) \ [a \vee (b \wedge c)] \wedge d = [(a \vee b) \vee c] \wedge d \).

\( (iv) \ a \leq b \) implies \( a \wedge c \leq b \wedge c, c \wedge a \leq c \wedge b \) and \( c \vee a \leq c \vee b \).

\( \text{Definition 2.5. } [7] \) An element \( 0 \in L \) is called zero element of \( L \), if \( 0 \wedge a = 0 \) for all \( a \in L \).

\( \text{Lemma 2.3. } [7] \) If \( L \) has 0, then for any \( a, b \in L \), we have the following:

\( (i) \ a \vee 0 = a, (ii) \ 0 \vee a = a \) and \( (iii) \ a \wedge 0 = 0 \).

\( \text{Definition 2.6. } [13] \) Let \( L \) be a non-empty set and \( x_0 \in L \). If for \( x, y \in L \) we define

\( x \land y = y \) if \( x \neq x_0 \)

\( x \land y = x \) if \( x = x_0 \) and

\( x \lor y = x \) if \( x \neq x_0 \)

\( x \lor y = y \) if \( x = x_0 \),

then \( (L, \lor, \land, x_0) \) is an ADL with \( x_0 \) as zero element. This is called discrete ADL with zero.

An element \( x \in L \) is called maximal if, for any \( y \in L \), \( x \leq y \) implies \( x = y \).

We immediately have the following.

\( \text{Lemma 2.4. } [7] \) For any \( m \in L \), the following are equivalent:

\( (1) \) \( m \) is maximal

\( (2) \) \( m \lor x = m \) for all \( x \in L \)

\( (3) \) \( m \land x = x \) for all \( x \in L \).

\( \text{Definition 2.7. } [7] \) A nonempty subset \( I \) of \( L \) is said to be an ideal if and only if it satisfies the following:
(1) \(a, b \in I \Rightarrow a \lor b \in I\)
(2) \(a \in I, \ x \in L \Rightarrow a \land x \in I\).

**Definition 2.8.** [7] A nonempty subset \(I\) of \(L\) is said to be an initial segment of \(L\) if, \(a \in L\) and \(x \in L\) such that \(x \leq a\) imply that \(x \in L\).

**Definition 2.9.** [10] A nonempty subset \(I\) of \(L\) is said to be a weak ideal if and only if it satisfies the following:
(1) \(a, b \in I \Rightarrow a \lor b \in I\)
(2) \(I\) is an initial segment of \(L\).

Observe that every ideal of \(L\) ia weak ideal, but not converse.

**Definition 2.10.** [7] A function \(f : L \rightarrow L\) is said to be an ADL homomorphism if it satisfies the following:
(1) \(f(x \land y) = fx \land fy\),
(2) \(f(x \lor y) = fx \lor fy\) for all \(x, y \in L\).

**Definition 2.11.** A function \(d : L \rightarrow L\) is called an isotone, if \(dx \leq dy\) for any \(x, y \in L\) with \(x \leq y\).

### 3. Permuting tri-\(f\)-derivations in ADLs.

We begin this paper with the following definition of a permuting map in an ADL.

**Definition 3.1.** [12]
(i) A map \(D : L \times L \times L \rightarrow L\) is called permuting map if
\[
D(x, y, z) = D(x, z, y) = D(y, z, x) = D(y, x, z) = D(z, x, y) = D(z, y, x)
\]
for all \(x, y, z \in L\).

(ii) \(D\) is called an isotone map if, for any \(x, y, z, w \in L\) with \(x \leq w\), \(D(x, y, z) \leq D(w, y, z)\).

(iii) The mapping \(d : L \rightarrow L\) defined by \(dx = D(x, x, x)\) for all \(x \in L\), is called the trace of \(D\).

**Definition 3.2.** [12] A permuting map \(D : L \times L \times L \rightarrow L\) is called a permuting tri-\(f\)-derivation on \(L\), if
\[
D(x \land w, y, z) = [w \land D(x, y, z)] \lor [x \land D(w, y, z)]
\]
for all \(x, y, z, w \in L\).

Now, the following definition gives the notion of permuting tri-\(f\)-derivation in an ADL.

**Definition 3.3.** A permuting map \(D : L \times L \times L \rightarrow L\) is called a permuting tri-\(f\)-derivation on \(L\), if there exists a function \(f : L \rightarrow L\) such that
\[
D(x \land w, y, z) = [fw \land D(x, y, z)] \lor [x \land D(w, y, z)]
\]
for all \(x, y, z, w \in L\).

Observe that a permuting tri-\(f\)-derivation \(D\) on \(L\) also satisfies
\[ D(x, y \wedge w, z) = [fw \wedge D(x, y, z)] \lor [fy \wedge D(x, w, z)] \] and
\[ D(x, y, z \wedge w) = [fw \wedge D(x, y, z)] \lor [fz \wedge D(x, y, w)] \]
for all \( x, y, z, w \in L \).

**Example 3.1.** Every permuting tri-derivation on \( L \) is a permuting tri-\( f \) -derivation, where \( f : L \to L \) is the identity map.

**Example 3.2.** Let \( L \) be an ADL with 0 and \( 0 \neq a \in L \). If we define a mapping \( D : L \times L \times L \to L \) by \( D(x, y, z) = a \) for all \( x, y, z \in L \) and \( f : L \to L \) by \( fx = a \) for all \( x \in L \), then \( D \) is a permuting tri-\( f \)-derivation on \( L \) but not a permuting tri-derivation on \( L \).

**Example 3.3.** Let \( L \) be an ADL with at least two elements. If we define a mapping \( D : L \times L \times L \to L \) by \( D(x, y, z) = (x \lor y) \lor z \), then \( D \) is not a permuting tri-\( f \)-derivation on \( L \) but, not a permuting tri-derivation on \( L \).

**Example 3.4.** Let \( L \) be an ADL with at least three elements and \( a \in L \). If we define the mapping \( D : L \times L \times L \to L \) by \( D(x, y, z) = (x \lor y \lor z) \lor a \) for all \( x, y, z \in L \) and \( f : L \to L \) by \( fx = a \) for all \( x \in L \), then \( D \) is a permuting tri-\( f \)-derivation on \( L \) but, not a permuting tri-derivation on \( L \).

**Lemma 3.1.** Let \( D \) be a permuting tri-\( f \)-derivation on \( L \). Then the following identities hold:

1. \( D(x, y, z) = fx \wedge D(x, y, z) = fy \wedge D(x, y, z) = fz \wedge D(x, y, z) \) for all \( x, y, z \in L \).
2. If \( L \) has 0 and \( f0 = 0 \), then \( D(0, y, z) = 0 \) for all \( y, z \in L \).
3. \( (fx \lor fy) \wedge D(x \wedge y, y, z) = D(x \wedge y, y, z) \) for all \( x, y, z, w \in L \).
4. \( fx \wedge dx = dx \) for all \( x \in L \).

**Proof.** Let \( x, y, z, w, w \in L \).

\( D(x, y, z) = D(x \wedge x, y, z) = [fx \wedge D(x, y, z)] \lor [fy \wedge D(x, y, z)] = fx \wedge D(x, y, z) \).

Similarly, \( fy \wedge D(x, y, z) = D(x, y, z) = fz \wedge D(x, y, z) \).

(2) Suppose \( L \) has 0 and \( f0 = 0 \). Now by (1) above, \( D(0, y, z) = f0 \wedge D(0, y, z) = 0 \wedge D(0, y, z) = 0 \).

(3) \( (fx \lor fx) \wedge D(x \wedge w, y, z) = (fx \lor fx) \wedge [[fw \wedge D(x, y, z)] \lor [fx \wedge D(w, y, z)] = [fw \wedge D(x, y, z)] \lor [fx \wedge D(w, y, z)] = D(x \wedge w, y, z) \).

(4) By (1) above, we get that \( fx \wedge D(x, x, x) = D(x, x, x) \). Thus \( fx \wedge dx = dx \).

**Theorem 3.1.** Let \( D \) be a permuting tri-\( f \)-derivation on \( L \) and \( m \) be a maximal element in \( L \). Then the following hold:

1. If \( x, y, z \in L \) such that \( fx \leq D(m, y, z) \), then \( D(x, y, z) = fx \).
2. If \( x, y, z \in L \) such that \( fx \geq D(m, y, z) \) and \( f \) \( m \) is a maximal element in \( L \), then \( D(x, y, z) \geq D(m, y, z) \).

**Proof.** (1) Let \( x, y, z \in L \) with \( fx \leq D(m, y, z) \). Then \( D(x, y, z) = D(m \wedge x, y, z) = [fx \wedge D(m, y, z)] \lor [fm \wedge D(x, y, z)] = fx \lor [fm \wedge D(x, y, z)] = (fx \lor fm) \wedge D(x, y, z) = (fx \lor fm) \wedge fx = fx \), by Lemma 3.1.

(2) Let \( x, y, z \in L \) with \( fx \geq D(m, y, z) \). Then \( D(x, y, z) = D(m \wedge x, y, z) = [fx \wedge D(m, y, z)] \lor [fm \wedge D(x, y, z)] = fx \lor [fm \wedge D(x, y, z)] = (fx \lor fm) \wedge D(x, y, z) = (fx \lor fm) \wedge fx = fx \), by Lemma 3.1.
[fx \land D(m, y, z)] \lor [fm \land D(x, y, z)] = D(m, y, z) \lor D(x, y, z). \text{ Thus } D(x, y, z) \geq D(m, y, z).

**Theorem 3.2.** Let D be a permuting tri-f-derivation on L where f is an increasing function on L. If x, y, z \in L such that w \leq x and D(x, y, z) = fx, then D(w, y, z) = fw.

**Proof.** Let x, y, z \in L with w \leq x and D(x, y, z) = fx. Since f is an increasing function on L, fw \leq fx. Now D(w, y, z) = D(x \land w, y, z) = [fw \land D(x, y, z)] \lor [fx \land D(w, y, z)] = [fw \land fx] \lor [fx \land fw \land D(w, y, z)] = fw \lor [fw \land D(w, y, z)] = fw.

**Theorem 3.3.** Let D be a permuting tri-f-derivation on L. Then for any x, y, z, w \in L, the following hold:

1. If D is an isotone map on L, then fx \land D(x \lor w, y, z) = D(x, y, z).
2. If f is either a join preserving or an increasing function on L, then fx \land D(x \lor w, y, z) \leq D(x, y, z).

**Proof.** Let x, y, z, w \in L.

1. Suppose D is an isotone map on L. Then D(x, y, z) \leq D(x \lor y, z). Now D(x, y, z) = D((x \land w) \lor x, y, z) = [fx \land D(x \lor w, y, z)] \lor [fx \land D(x, y, z)] = [fx \land D(x \lor w, y, z)] \lor [fx \land D(x \lor w, y, z)] = [fx \land D(x \lor w, y, z)] \lor [fx \land D(x \lor w, y, z)] = [fx \land D(x \lor w, y, z)] \lor [fx \land D(x \lor w, y, z)] = [fx \land D(x \lor w, y, z)] \lor D(x, y, z).

2. Suppose f is an increasing function on L. Then fx \leq f(x \lor w). Now D(x, y, z) = D((x \lor w) \land x, y, z) = [fx \land D(x \lor w, y, z)] \lor [fx \land D(x, y, z)] = [fx \land D(x \lor w, y, z)] \lor [fx \land D(x, y, z)] = [fx \land D(x \lor w, y, z)] \lor D(x, y, z).

**Theorem 3.4.** Let D be a permuting tri-f-derivation on L and m be a maximal element in L. If f is a homomorphism on L, then the following are equivalent:

1. D is an isotone map on L
2. D(x, y, z) = fx \land D(m, y, z) for all x, y, z \in L
3. D is a join preserving map on L
4. D is a meet preserving map on L.

**Proof.** Let f be a homomorphism on L and x, y, z \in L.

1. \Rightarrow 2 : D(x, y, z) = D(m \land x, y, z) = [fx \land D(m, y, z)] \lor [fm \land D(x, y, z)]. Thus fx \land D(m, y, z) \leq D(x, y, z). On the other hand, fx \land D(x \land m, y, z) = fx \land [[fm \land D(x, y, z)] \lor [fx \land D(m, y, z)]] = [fx \land fm \land D(x, y, z)] \lor [fx \land D(m, y, z)] = [fm \land fx \land D(x, y, z)] \lor [fx \land D(m, y, z)] = [fx \land D(x, y, z)] \lor [fx \land D(m, y, z)] = D(x, y, z) \lor D(m, y, z). Since D is an isotone map on L, D(x \land m, y, z) \leq D(m, y, z). Thus D(x, y, z) = fx \land D(x \land m, y, z) \leq fx \land D(m, y, z). Hence
\[ D(x, y, z) = fx \land D(m, y, z). \]

(2) \( \Rightarrow \) (3) : \( D(x \lor w, y, z) = (f(x \lor w) \land D(m, y, z) = (fx \land D(m, y, z)) \lor (fy \land D(m, y, z)) = D(x, y, z) \lor D(w, y, z). \) Thus \( D \) is a join preserving map on \( L \).

(2) \( \Rightarrow \) (4) : \( D(x \land w, y, z) = (f(x \land w) \land D(m, y, z) = fx \land f(w \land D(m, y, z) = D(x, y, z) \land D(w, y, z). \) Thus \( D \) is a meet preserving map on \( L \).

(3) \( \Rightarrow \) (1) and (4) \( \Rightarrow \) (1) are trivial.

Theorem 3.5. Let \( d \) be the trace of the permuting tri-f-derivation \( D \) on an associative ADL \( L \). Then \( d(x \land y) = (fy \land dx) \lor D(x, y, y) \lor (fx \land dy) \) for all \( x, y, z \in L \).

Proof. Let \( x, y, z \in L \). Then
\[
fy \land D(x, x \land y, x \land y) = fy \land [fy \land D(x, x, x \land y)] \lor [fx \land D(x, y, x \land y)] = \]

\[
[fy \land D(x, x, x \land y)] \lor D(x, y, x \land y) = [fy \land [fy \land D(x, x, x)] \lor [fx \land D(x, x, y)]] \lor \]

\[
[[fy \land D(x, x, x)] \lor [fx \land D(x, x, y)]] = (fy \land dx) \lor D(x, x, x) \lor (fy \land dy). \]

Again, \( fx \land D(y, x \land y, x \land y) = fx \land [fy \land D(y, y, x \land y)] \lor [fy \land D(y, x, x \land y)] = D(y, x, x \land y) \lor [fx \land D(y, y, x \land y)] = [fy \land D(y, x, x)] \lor [fx \land D(y, x, y)] \lor [fy \land D(y, x, y)] \lor (fx \land dy). \]

Thus \( d(x \land y) = D(x \land y, x \land y, x \land y) = (fy \land D(x, x \land y, x \land y)] \lor [fx \land D(y, x, x \land y)] \lor (fx \land dy). \) \( \square \)

Theorem 3.6. Let \( d \) be the trace of the join preserving permuting tri-f-derivation \( D \) on an associative ADL \( L \). If \( f \) is a join preserving map on \( L \), then \( F_d(L) = \{x \in L | dx = fx\} \) is a weak ideal in \( L \).

Proof. Suppose \( f \) is a join preserving map on \( L \). Let \( x \in L, y \in F_d(L) \) and \( x \leq y \). Since \( f \) is a join preserving, \( f \) is an increasing function on \( L \) and hence \( fx \leq fy \). Now, by Theorem 3.5,
\[
dx = d(y \land x) = (fx \land dy) \lor D(y, y, x) \lor (fy \land dx) = fx \lor D(y, y, x) \lor \]

\[
D(y, x, x) \lor (fy \land dx) = fx \lor (fy \land dx) = fy \land fx = fx. \) Thus \( x \in F_d(L). \)

Let \( x, y \in F_d(L) \). Then \( d(x \lor y) = D(x \lor y, x \lor y, x \lor y) = D(x, x \lor y, x \lor y) \lor D(y, x \lor y, x \lor y) \lor D(y, x, x \lor y) \lor D(y, x, x) \lor D(y, y, x) \lor D(y, x, y) \lor D(y, y, x) \lor (dy = \)

\[
dx \lor D(x, y) \lor D(x, y) \lor D(x, x) \lor D(x, y) \lor (dy = fx \lor D(x, y) \lor D(y, x, y) \lor (dy = fx \lor D(x, y) \lor D(y, x, y) \lor (dy = fx \lor D(x, y) \lor D(x, y) \lor (dy = fx \lor D(x, y) \lor D(x, y) \lor (dy = fx \lor D(x, y) \lor D(x, y) \lor (dy = fx \lor D(x, y) \lor D(x, y) \lor (dy = fx \lor D(x, y) \lor D(x, y) \lor (dy = fx \lor D(x, y) \lor D(x, y) \lor (dy = fx \lor D(x, y) \lor D(x, y) \lor (dy = fx \lor D(x, y) \lor D(x, y) \lor (dy = fx \lor D(x, y) \lor D(x, y) \lor (dy = fx \lor D(x, y) \lor D(x, y) \lor (dy = fx \lor D(x, y) \lor D(x, y) \lor (dy = fx \lor D(x, y) \lor D(x, y) \lor (dy = fx \lor D(x, y) \lor D(x, y) \lor (dy = \)
Therefore $D(x, x, y) = 0$ for all $x, y \in L$. In particular, $D(x \lor z, x \lor z, y) = 0$ and hence $D(x, y, z) = 0$. Therefore $D = 0$. 

Let us recall the definition of a prime ADL in the following.

**Definition 3.4.** [12] An ADL $L$ with $0$ is said to be a prime ADL if, for $a, b \in L$, $a \land b = 0$ implies either $a = 0$ or $b = 0$.

**Theorem 3.7.** Let $L$ be an associative prime ADL and $d_1, d_2$ be the traces of join preserving permuting tri-$f_1$, tri-$f_2$-derivations $D_1, D_2$ on $L$, respectively. If $d_1x \land d_2x = 0$ for all $x \in L$, then either $D_1 = 0$ or $D_2 = 0$.

**Proof.** Suppose $d_1x \land d_2x = 0$ for all $x \in L$. Assume that $d_1 \neq 0$ and $d_2 \neq 0$. Then $d_1y \neq 0$ and $d_2z \neq 0$ for some $y, z \in L$. Now, $d_1(y \lor z) = D_1(y \lor z, y \lor z, y \lor z) = d_1y \lor D_1(y, y, z) \lor D_1(y, z, z) \lor d_1z \neq 0$ and $d_2(y \lor z) = D_2(y \lor z, y \lor z, y \lor z) = d_2y \lor D_2(y, y, z) \lor D_2(y, z, z) \lor d_2z \neq 0$. But, by our assumption $d_1(y \lor z) \land d_2(y \lor z) = 0$. This is a contradiction, (since $L$ is a prime ADL). Thus $d_1 = 0$ or $d_2 = 0$ and hence by Lemma 3.2, either $D_1 = 0$ or $D_2 = 0$. 

Finally we conclude this paper with the following theorem.

**Theorem 3.8.** Let $L$ be an associative prime ADL and $d_1, d_2$ be the traces of join preserving permuting tri-$f_1$, tri-$f_2$-derivations $D_1, D_2$ on $L$, respectively such that $d_1f_2 = d_1$ and $f_1d_2 = d_2$. Suppose one of the following condition hold

1. $D_1(d_2x, f_2x, f_2x) = 0$ for all $x \in L$
2. $D_1(d_2x, d_2x, f_2x) = 0$ for all $x \in L$
3. $d_1od_2 = 0$, then either $D_1 = 0$ or $D_2 = 0$.

**Proof.** (1) Suppose $D_1(d_2x, f_2x, f_2x) = 0$ for all $x \in L$. Let $x \in L$. Since $f_2x \land d_2x = d_2x$, we get that $[f_1(d_2x) \land D_1(f_2x, f_2x, f_2x)] \lor [f_1(f_2x) \land D_1(d_2x, f_2x, f_2x)] = D_1(f_2x \land d_2x, f_2x, f_2x) = 0$. Thus $(f_1od_2x) \land (d_1of_2x) = 0$. Therefore $d_2x \land d_1x = 0$.

(2) Suppose $D_1(d_2x, d_2x, f_2x) = 0$ for all $x \in L$. Let $x \in L$. Again since $f_2x \land d_2x = d_2x$, we get that $[f_1(d_2x) \land D_1(f_2x, d_2x, f_2x)] \lor [f_1(f_2x) \land D_1(d_2x, d_2x, f_2x)] = D_1(f_2x \land d_2x, d_2x, f_2x) = 0$. Thus $(f_1od_2x) \land D_1(f_2x, d_2x, f_2x) = 0$. Therefore $d_2x \land D_1(f_2x, d_2x, f_2x) = 0$. Thus $d_2x \land D_1(f_2x, f_2x, f_2x) = 0$. Therefore $d_2x \land (f_1od_2x) \land \land (d_1of_2x) = 0$. Therefore $d_2x \land (f_1od_2x) \land d_1x = 0$.

(3) Suppose $d_1od_2 = 0$. Then $d_1(d_2x) = 0$ for all $x \in L$. So that, $D_1(d_2x, d_2x, d_2x) = 0$ for all $x \in L$. Let $x \in L$. Again since $f_2x \land d_2x = d_2x$, we get that $[f_1(d_2x) \land D_1(d_2x, d_2x, f_2x)] \lor [f_1(f_2x) \land D_1(d_2x, d_2x, d_2x)] = D_1(d_2x, d_2x, f_2x \land d_2x) = 0$. Therefore $d_2x \land D_1(d_2x, d_2x, f_2x) = 0$. Thus $[d_2x \land D_1(d_2x, f_2x, f_2x) \lor [d_2x \land f_1(d_2x) \land D_1(d_2x, f_2x, f_2x)] = d_2x \land D_1(d_2x, f_2x \land d_2x, f_2x) = 0$. Hence $d_2x \land D_1(d_2x, f_2x, f_2x) = 0$. So that $d_2x \land D_1(d_2x, f_2x, f_2x) = 0$ and hence $d_2x \land d_1x = 0$. Therefore, $d_2x \land d_1x = 0$ for all $x \in L$ in all three cases. By Theorem 3.7, we get that either $D_1 = 0$ or $D_2 = 0$. 

□
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