THE COMMUTATIVITY OF PRIME NEAR RINGS

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Abstract

Let N be a near-ring, and σ be an automorphisms of N. An additive mapping d from a near-ring N into itself is called a reverse σ-derivation on N if d(xy) = d(y)x + σ(y)d(x), holds for all x, y ∈ N. In this paper, we shall investigate the commutativity of N by a reverse σ-derivation d satisfied some properties, when N is a prime ring.

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1. Introduction

Near-rings are one of the generalize structures of rings. A near-ring N is a ring (N, +, .), where + is not necessarily abelian and with only one distributive law. A left near-ring (resp. right near-ring) is called a zero-symmetric left near-ring (resp. a zero-symmetric right near-ring) if 0x = 0 (resp. x0 = 0), for all x ∈ N. A near-ring N called a prime near-ring if xNy = 0 implies x = 0 or y = 0, for all x, y ∈ N. The multiplicative center Z of N will denote, Z = { x ∈ N: xy = yx for all y ∈ N }. The symbol [x,y] will denote the commutator xy – yx, for all x, y ∈ N, and note that important identities [x, yz] = y[x,z] + [x,y]z and [xy, z] = x[y, z] + [x, z]y satisfied for all x, y, z ∈ N. An additive mapping d: N → N is called derivation if d(xy) = xd(y) + d(x)y, or equivalently (cf.[7]) that d(xy) = d(x)y + xd(y), for all x, y ∈ N. The derivation d will called commuting if d([x,y]) = 0, for all x ∈ N. The study of commutativity of prime near-rings by using derivations was initiated by H. E. Bell and G. Mason in 1987 [2], and Yilun Shang [8] satisfying the commutativity of prime near rings N if there exist k, l ∈ N such that N admits a generalized derivation D satisfying either D([x,y]) = xk [x,y]xk for all x, y ∈ N or D([x,y]) = -xk [x,y]xl for all x, y ∈ N.In [5] A. A. M. Kamal generalizes some results of Bell and Mason by studying the commutativity of 3-prime near-rings using a σ-derivation instead of the usual derivation, where σ is an automorphism on the near-ring. Bresar and Vukman in 1989 [3] have introduced the notion of a reverse derivation as an additive mapping d from a ring R into itself satisfying d(xy) = d(y)x + yd(x), for all x, y ∈ R. Samman and Alyamani [6] studied the reverse derivations on semi prime
rings. C.Jaya S. R., G.Venkata B.Rao and S.Vasantha Kumar in [4] studied generalized reverse derivation of a semi prime ring $R$ and proved that if $f$ is a generalized reverse derivation with a derivation $d$, then $f$ is a strong commutativity preserving and $R$ is commutative. Afrah M.Ibraheem in [1] used the notion of reverse derivations on a prime $\Gamma$-near ring $M$ to study the commutativity conditions of $M$, when $U$ be a non-zero invariant subset of $M$. In this paper, we shall prove that a prime near-ring which admits a nonzero reverse $\sigma$-derivation satisfying certain conditions must be a commutative ring. Throughout the paper $N$ will denote a zero symmetric near-ring with multiplicative center $Z$.

2. Preliminary Results

To prove our results we start with the following definition and lemmas:

**Definition 2.1:**

Let $N$ be a near-ring, and $\sigma$ is an automorphism on $N$. An additive mapping $d$ from $N$ into itself is called a reverse $\sigma$-derivation on $N$ if satisfying $d(xy) = d(y)x + \sigma(y)d(x)$, for all $x, y \in N$.

**Lemma 2.2:**

Let $d$ be an arbitrary additive automorphism of $N$. Then $d(xy) = \sigma(y)d(x)+d(y)x$ for all $x, y \in N$ if and only if $d(xy) = d(y)x+\sigma(y)d(x)$, for all $x, y \in N$. Therefore $d$ is a reverse $\sigma$-derivation if and only if $d(xy) = d(y)x+\sigma(y)d(x)$.

**Proof:** Suppose

$$d(xy) = \sigma(y)d(x)+d(y)x,$$

for all $x, y \in N$. Since

$$(x+x)y = xy + xy,$$

$$d((x+x)y) = d(xy + xy)$$

$$d((x+x)y)) = \sigma(y)d(x+x)+ d(y)(x+x).$$

$$= \sigma(y)d(x)+ \sigma(y)d(x)+ d(y)x + d(y)x...$$

$$= \sigma(y)d(x)+ \sigma(y)d(x)+ d(y)x + d(y)x...$$

for all $x, y \in N$. And,

$$d(xy+xy) = d(xy)+ d(xy)$$

$$= \sigma(y)d(x)+ d(y)x+ \sigma(y)d(x)+ d(y)x...$$

for all $x, y \in N$. From (1) and (2), we get

$$\sigma(y)d(x)+ d(y)x = d(y)x+ \sigma(y)d(x),$$

So,

$$d(xy) = d(y)x+ \sigma(y)d(x),$$

for all $x, y \in N$. The converse is similarly.
**Lemma 2.3:**

Let $N$ be a prime near-ring, and $d$ be a nonzero reverse $\sigma$-derivation of $N$. If $d(N) \subseteq Z(N)$ then $N$ is a commutative ring.

**Proof**

Let $d(x) \in Z(N)$, for all $x \in N$. Then

$$d(x)z = zd(x) \ldots (1)$$

Replacing $x$ by $xy$ in (1), we have

$$(d(y)x + \sigma(y)d(x))z = z(d(y)x + \sigma(y)d(x)),$$

Then

$$\sigma(y)d(x)z - zd(y)x = -d(y)xz + zd(y)x \ldots (2),$$

for all $x, y \in N$. Replacing $\sigma(y)$ by $d(x)$ in (2) and using (1), we get

$$d(y)z(xz + z) = d(y)[x, z] = 0 \ldots (3),$$

for all $x, y, z \in N$. Replacing $z$ by $zy$ in (3) and using (3) again, we get

$$d(y)z[y, x] = 0,$$

for all $x, y, z \in N$. Since $N$ is a prime, and $d \neq 0$, we have

$$[y, x] = 0,$$

for all $x, y \in N$. Therefore $N$ is commutative.

**Lemma 2.4:**

Let $N$ be a prime near-ring with center $Z$, and let $d$ be a nonzero reverse $\sigma$-derivation of $N$, then $d(Z) \subseteq Z$.

**Proof:** For any $z \in Z$ and $x \in N$, we have

$$d(zx) = d(zx),$$

$$d(zx) = d(z)x + \sigma(z)d(x)$$

$$= \sigma(z)d(x) + d(z)x,$$

by lemma 2.2. If we replace $\sigma(z)$ by $z$, we get

$$d(zx) = zd(x) + d(z)x \ldots (1),$$

for all $x, z \in N$.

$$d(zx) = d(xz) + \sigma(x)d(z) \ldots (2),$$

for all $x, z \in N$. From (1) and (2), we get

$$d(z)x = \sigma(x)d(z),$$
and since $\sigma$ is automorphism, we have
\[ d(z)x = xd(z), \]
for all $x, z \in N$. Therefore $d(z) \in Z$, this complete proof.

**Lemma 2.5:**

Let $d$ be a nonzero reverse $\sigma$-derivation of a prime near-ring $N$, and $x \in N$. If $xd(N) = 0$ or $d(N)x = 0$, then $x = 0$.

**Proof:** Let assume that,
\[ x \ d(n) = 0 \ldots \tag{1} \]
for all $n \in N$. Replacing $n$ by $mn$ in (1), we have
\[ x \ d(n)m + x \ \sigma(n)d(m) = 0 \ldots \tag{2} \]
for all $x, n, m \in N$. By using (1) in (2), and since $\sigma$ is automorphism, we have
\[ x \ N \ d(m) = 0, \]
for all $x, m \in N$, and since $N$ is a prime, and $d(N) \neq 0$, we have $x = 0$.

Similarly, we can prove $x = 0$, if $d(N)x = 0$.

3. **The Commutativity of Prime Near Ring $N$**

In this section we give conditions under which a prime near ring $N$ must be commutative ring.

**Theorem 3.1:**

For a prime near ring $N$, let $d$ be a nonzero reverse $\sigma$-derivation of $N$, such that $[x, d(x)] = 0$, for all $x \in N$, then $N$ is commutative.

**Proof:** Let
\[ [x, d(x)] = 0 \ldots \tag{1} \]
for all $x \in N$. Replacing $d(x)$ by $yd(x)$ in (1) and using (1) again, we have
\[ [x, y] \ d(x) = 0 \ldots \tag{2} \]
for all $x, y \in N$. Replace $y$ by $zy$ in equ.(2) and using (2), we get,
\[ [x ,z] \ y \ d(x) = 0, \]
for all $x, y, z \in N$. Since $N$ is a prime, we have either $[x, z]=0$ or $d(x)= 0$.

Since $d(x)\neq 0$, for all $x \in N$, then we have $[x, z]= 0$, it follows that $x \in Z(N)$ for each fixed $x \in N$, and by lemma 2.4, we get $d(x) \in Z(N)$, that's $d(N) \subset Z(N)$. Then by lemma 2.3, we get $N$ is commutative.
Theorem 3.2:

Let $N$ be a prime near ring, and $d$ be a nonzero reverse $\sigma$-derivation of $N$. If $[d(y), d(x)] = 0$, for all $x, y \in N$, then $N$ is commutative.

Proof: Given that

$$[d(y), d(x)] = 0 \ldots \quad (1),$$

for all $x, y \in N$. Replacing $y$ by $yx$ in (1), we get,

$$[d(xy) + \sigma(x)d(y), d(x)] = 0,$$

By using (1) again, we get

$$d(x)[y, d(x)] + [\sigma(x), d(x)] d(y) = 0 \ldots \quad (2),$$

for all $x, y \in N$. Replacing $y$ by $zy$, where $z \in Z(N)$ in equ.(2), we get,

$$d(x)[y, d(x)] + d(x)[z, d(x)] y + [\sigma(x), d(x)] d(y) z + [\sigma(x), d(x)] \sigma(y) d(z) = 0 \ldots \quad (3),$$

for all $x, y, z \in N$. Since $\sigma$ is automorphism, and by using (2) in (3), we get

$$[\sigma(x), d(x)] y d(z) = 0,$$

for all $x, y, z \in N$. Since $N$ is a prime, we have either

$$[\sigma(x), d(x)] = 0, \text{ or } d(z) = 0.$$

Since $d(z) \neq 0$, we have

$$[\sigma(x), d(x)] = 0 \ldots \quad (4),$$

for all $x \in N$. Replacing $\sigma(x)$ by $x$ in (4), and by using the similar procedure as in Theorem 3.1, we get, $N$ is commutative.

Theorem 3.3:

Let $N$ be a prime near ring, and $d$ be a nonzero reverse $\sigma$-derivation of $N$. If $[x, d(y)] \in Z(N)$, for all $x, y \in N$, then $N$ is commutative.

Proof: Assume that

$$[x, d(y)] \in Z(N),$$

for all $x, y \in N$. Hence for all $n \in N$,

$$[[x, d(y)], n] = 0 \ldots \quad (1).$$

Replacing $x$ by $xd(y)$ in (1), and using (1) again, we get

$$[x, d(y)] [d(y), n] = 0 \ldots \quad (2),$$

for all $x, y, n \in N$. 
Replacing $x$ by $nx$ in (2), and using (2) again, we get
\[ [n, d(y)] x [d(y), n] = 0... \quad (3), \]
for all $x, y, n \in N$. Since $N$ is a prime, we have either
\[ [n, d(y)] = 0... \quad (4), \]
for all $y, n \in N$, or
\[ [d(y), n] = 0... \quad (5), \]
for all $y, n \in N$. If we replacing $d(y)$ by $md(y)$ in (4) and (5), and using them again, we get
\[ [n, m] d(y) = 0 \]
or
\[ [m, n] d(y) = 0, \]
for all $y, n, m \in N$. By using lemma 2.5 in two cases, we have
\[ [n, m] = 0, \text{ and } [m, n] = 0, \]
for all $n, m \in N$. Therefore, $N$ is commutative.

**Theorem 3.4:**

Let $N$ be a prime near ring, $d$ be a nonzero reverse $\sigma$-derivation of $N$, and $y \in N$. If \([d(x), y] = 0\) then $d(y) = 0$ or $y \in Z(N)$.

**Proof:** Let
\[ [x, d(x)] = 0... \quad (1), \]
for all $x \in N$. Replacing $d(x)$ by $yd(x)$ in (1) and using (1) again, we have
\[ [x, y] d(x) = 0... \quad (2), \]
For all $x, y \in N$. Replace $y$ by $zy$ in equ.(2) and using (2), we get,
\[ [x, z] y d(x) = 0, \]
For all $x, y, z \in N$. Since $N$ is a prime, we have either
\[ [x, z] = 0 \] or $d(x) = 0$.

Since $d(x) \neq 0$, for all $x \in N$, then we have \([x, z] = 0\), it follows that $x \in Z(N)$ for each fixed $x \in N$, and by lemma 2.4, we get $d(x) \in Z(N)$, that's $d(N) \subseteq Z(N)$. Then by lemma 2.3, we get $N$ is commutative.

**Theorem 3.5:**

Let $N$ be a prime near ring, and $d$ be a nonzero reverse $\sigma$-derivation of $N$, such that $d([x, y]) = [x, d(y)]$, for all $x, y \in N$, then $N$ is commutative.
Proof: Given that
\[ d([x, y]) = [x, d(y)] \ldots \]  
(1),

for all \( x, y \in N \). Replacing \( y \) by \( yx \) in (1) and using (1), we get,
\[ [x, d(x)] y + [x, \sigma(x)] d(y) = 0 \ldots \]  
(2),

for all \( x, y \in N \). If we replacing \( \sigma(x) \) by \( x \) in (2), we have
\[ [x, d(x)] y = 0 \ldots \]  
(3),

for all \( x, y \in N \). Replacing \( y \) by \( yd(x) \) in (3), we get
\[ [x, d(x)] y d(x) = 0, \]  
for all \( x, y \in N \). Since \( N \) is a prime, and \( d \neq 0 \), we have
\[ [x, d(x)] = 0, \]  
for all \( x \in N \). Then by theorem 3.1, we get, \( N \) is commutative.

4. Conclusions

For an automorphism \( \sigma \) on a near ring \( N \), we study the commutativity on \( N \), if \( N \) has a non zero reverse \( \sigma \)-derivation \( d \), where \( d \) is defined as an additive mapping from \( N \) into itself satisfying
\[ d(xy) = d(y)x + \sigma(y)d(x), \]  
for all \( x, y \in N \), and introduced some conditions on \( d \) to get the commutativity on \( N \) when \( N \) is a prime near ring.

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References


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