Certain results on Ricci solitons in $\alpha$-Kenmotsu manifolds

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In this paper, we study some curvature problems of Ricci solitons in $\alpha$-Kenmotsu manifold. It is shown that a symmetric parallel second order-covariant tensor in a $\alpha$-Kenmotsu manifold is a constant multiple of the metric tensor. Using this result, it is shown that if $(L_v g + 2 S)$ is parallel where $V$ is a given vector field, then the structure $(g, V, \lambda)$ yield a Ricci soliton. Further, by virtue of this result, Ricci solitons for $n$-dimensional $\alpha$-Kenmotsu manifolds are obtained. In the last section, we discuss Ricci soliton for 3-dimensional $\alpha$-Kenmotsu manifolds.

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Introduction

A Ricci soliton are the natural generalization of Einstein metric and are defined on a Riemannian manifold. On the manifold $\mathcal{M}$, a Ricci soliton is a triple $(g, V, \lambda)$ with a Riemannian metric $g$, a vector field $V$ and a real scalar $\lambda$ such that

$$(L_V g)(X, Y) + 2 S(X, Y) + 2\lambda g(X, Y) = 0, \quad \ldots (1)$$

for any vector fields $X, Y$ on $\mathcal{M}$ where $S$ is the Ricci tensor and $L_V$ denotes the Lie derivative operator along the vector field $V$. The metric satisfying (1) are very interesting in the field of physics and are often referred as quasi-Einstein. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda$ is negative, zero and positive respectively.

Das$^4$ studied second order parallel tensor on an almost contact metric manifold and found that on an $\alpha$-K-contact manifold ($\alpha$ being non-zero real constant) a second order symmetric parallel tensor is a constant multiple of the associative positive definite Riemannian metric tensor. It is also proved that in an $\alpha$-Sasakian manifold there is no non-zero parallel 2-form. The study of Ricci solitons in K-contact manifolds was started by Sharma$^5$ and in the continuation of this Ghosh, Sharma and Cho$^7$ studied gradient Ricci soliton of a non-Sasakian $(k,\mu)$ -contact manifold. Generally, in a P-Sasakian manifold the structure vector field $\xi$ is not killing, that is $(L_\xi g) \neq 0$ but in K-contact manifold $\xi$ is a killing vector field, that is $(L_\xi g) = 0$. Recently, De$^9$ have studied Ricci soliton in P-Sasakian, Barua and De$^{10}$ have studied Ricci soliton in Riemannian manifolds. Since then several other studied Ricci soliton have been published in various contact manifolds. Eisenhart problem to Ricci soliton in $f$-Kenmotsu manifold, Eta-Ricci solitons on para-Kenmotsu manifolds, on contact and Lorentzian manifolds on Sasakian manifold, $\alpha$ -Sasakian manifold, $\alpha$-Kenmotsu manifold, etc.

Motivated by above studies, in this paper we treat Ricci soliton in $\alpha$-Kenmotsu manifolds. The paper is structured as follows. After
introduction, section 2 is a brief review of α-Kenmotsu manifold. Section 3, is devoted to the study of parallel symmetric second order tensor in α-Kenmotsu manifold and Ricci soliton in α-Kenmotsu manifolds. In this section, we obtain relation between symmetric parallel second order covariant tensor and metric tensor in α-Kenmotsu manifold. In the second problem of this section we studied the necessary and sufficient condition of a Ricci semi-symmetric α-Kenmotsu manifold and n-Einstein manifold Section 4 is devoted to study Ricci soliton in 3-dimensional α-Kenmotsu manifold.

α-Kenmotsu manifold

An n-dimensional real $C^\omega$-manifold $M$ is said to almost contact structure $(\varphi, \xi, \eta)$ if it admits a $(1, 1)$ tensor field $\varphi$, a contravariant vector field $\xi$ and a 1-form $\eta$ which satisfy

\[ \eta(\xi) = 1, \quad \varphi^X = -X + \eta(X)\xi, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 0, \]

for all vector field $X, Y$ on $\chi(M)$, where $\chi(M)$ is the Lie algebra of $C^\omega$-vector fields on $M$. An n-dimensional real $C^\omega$-manifold $M$ equipped with almost contact structure $(\varphi, \xi, \eta)$ is called almost contact manifold. An almost contact manifold $M$ with metric tensor $g$ which satisfies the condition

\[ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \]

and

\[ g(X, \xi) = \eta(X), \]

is called almost contact metric manifold $M = (\varphi, \xi, \eta, g)$. An almost contact metric manifold $M$ is said to be almost $\alpha$-Kenmotsu manifold if

\[ d\varphi = 0, \quad d\xi = 2\alpha \eta \wedge \varphi, \]

where $\varphi$ is a fundamental 2-form defined as $\varphi(X, Y) = g(\varphi X, \varphi Y)$ and $\alpha$ being a non-zero real constant. Moreover, if an almost $\alpha$-Kenmotsu manifold $M$ satisfies the following relations

\[ \{\varphi g\}Y = -a(g(\varphi X, \varphi Y)\xi + \eta(Y)\varphi X), \]

and

\[ \{\varphi \xi\} = \alpha X - \eta(X)\xi, \]

then it is called $\alpha$-Kenmotsu manifold. On an $\alpha$-Kenmotsu manifold $M$, the following relations hold

\[ R(X, Y)\xi = \alpha^2(\eta(X)Y - \eta(Y)X), \]

\[ R(\xi, Y)X = \alpha^2(\eta(X)Y - g(X, Y)\xi), \]

\[ \eta(R(X, Y)Z) = \alpha^2(g(X, Y)\eta(Z) - g(Y, Z)\eta(X)), \]

\[ S(X, \xi) = -\alpha^2(n - 1)\eta(X), \]

\[ S(\xi, \xi) = -\alpha^2(n - 1), \]

\[ \mathcal{Q}_X = -\alpha^2(n - 1)\xi, \]

\[ \{\mathcal{Q}_X, \eta\}Y = a(g(X, Y) - \eta(X)\eta(Y)), \]

for all vector fields $X, Y, Z$ on $\chi(M)$, where $R$ is the Riemannian curvature tensor, $S$ is the Ricci tensor of type $(0, 2)$ and $\mathcal{Q}$ is the Ricci operator defined as $S(X, Y) = g(\mathcal{Q}X, Y)$.

Parallel symmetric second order tensors and Ricci solitons in $\alpha$-Kenmotsu manifolds

Let $h$ denote a $(0, 2)$ type symmetric tensor field which is parallel with respect to $\nabla$ that is $\nabla h = 0$. Then it follows that

\[ V^2 h(X, Y, Z, W) - V^2 h(X, Y; W, Z) = 0, \]

which gives

\[ h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0. \]

Taking $Z = W = \xi$ in (16) and using (8), we have

\[ \alpha^2(\eta(X)h(Y, \xi) - \eta(Y)h(X, \xi)) = 0. \]

Since $a$ is non-zero, so by taking $X = \xi$ in (17) and by the symmetry of $h$, we have

\[ h(Y, \xi) = \eta(Y)h(\xi, \xi). \]

Differentiating (18) covariantly with respect to $X$, we have

\[ (\nabla_X h)(Y, \xi) + h(\nabla_Y h)(Y, \xi) + h(Y, \nabla_X \xi) = \eta(\nabla_Y h)(\xi, \xi) + \eta(\nabla_X h)(\xi, \xi) + 2\eta(Y)h(\nabla_X \xi). \]

By using (7), (14), (18) and the parallel condition $\nabla h = 0$ in (19), we have

\[ h(X, Y) = g(X, Y)h(\xi, \xi). \]

The above equation implies that $h(\xi, \xi)$ is a constant, via (18). So we have the following theorem.

Theorem 1. A symmetric parallel second order covariant tensor in an $\alpha$-Kenmotsu manifold is a constant multiple of the metric tensor.

Corollary 1. A locally Ricci symmetric $(\nabla S = 0)$ $\alpha$-Kenmotsu manifold is an Einstein manifold.

Remark 1. The following statements for $\alpha$-Kenmotsu manifold are equivalent

(i) $\nabla h = 0$,

(ii) locally Ricci symmetric,

(iii) Ricci semi-symmetric, that is $R \cdot S = 0$.

The implication (i) $\rightarrow$ (ii) $\rightarrow$ (iii) is trivial. Now we prove that the implication (ii) $\rightarrow$ (i) in more general frame work of $\alpha$-Kenmotsu manifold. Since $R \cdot S = 0$, means exactly (16) with $h$
replaced by $S$, that is

$$(R(X,Y) \cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V). \quad \ldots (21)$$

Taking $R \cdot S = 0$ and putting $X = \xi$ in (21), we have

$$S(R(\xi,Y)U,V) + S(U,R(\xi,Y)V) = 0. \quad \ldots (22)$$

In view of (9) and $\alpha = 0$, the above equation becomes

$$\eta(U)S(Y,V) - g(Y,V)S(\xi,V) + \eta(V)S(U,Y) - g(Y,V)S(U,\xi) = 0. \quad \ldots (23)$$

Putting $U = \xi$ in (23) and by using (9), (11) and (12), we obtain

$$S(Y,V) = -\alpha^2(n-1)g(Y,V). \quad \ldots (24)$$

This lead the following theorem.

**Theorem 2.** A Ricci semi-symmetric $\alpha$-Kenmotsu manifold is an Einstein manifold.

**Corollary 2.** If on an $\alpha$-Kenmotsu manifold the tensor field $(\mathcal{L}_g + 2S)$ is parallel, then $(g,V,\lambda)$ gives a Ricci soliton.

**Proof.** A Ricci soliton in $\alpha$-Kenmotsu manifold is defined by (1). Thus $(\mathcal{L}_g + 2S)$ is parallel. By theorem (1) it is clear that if an $\alpha$-Kenmotsu manifold admits a symmetric parallel (0, 2) tensor, then the tensor is a constant multiple of the metric tensor. Hence $(\mathcal{L}_g + 2S)$ is a constant multiple of metric tensor $g$ that is $(\mathcal{L}_g + 2S)(X,Y) = g(X,Y)h(\xi,\xi)$, where $h(\xi,\xi)$ is a non zero constant. It is the application of the theorem (1) to Ricci soliton.

**Theorem 3.** If a metric $g$ in an $\alpha$-Kenmotsu manifold is a Ricci soliton with $V = \xi$ then it is $\eta$-Einstein.

**Proof.** Putting $V = \xi$ in (1), we have

$$(\mathcal{L}_g)(X,Y) + 2S(X,Y) + 2g(X,Y) = 0, \quad \ldots (25)$$

where $(\mathcal{L}_g)(X,Y) = g(P_VX,Y) + g(X,P_VY).$

Substituting (25) in (24) and by use of (7), we obtain

$$S(X,Y) = -(\alpha + \lambda)g(X,Y) + \alpha \eta(X)\eta(Y). \quad \ldots (26)$$

Hence the result.

**Theorem 4.** A Ricci soliton $(\xi, g, \lambda)$ in an $n$-dimensional $\alpha$-Kenmotsu manifold can not be steady but is shrinking.

**Proof.** In the Linear Algebra either the vector field $V \in \text{Span} \xi$ or $V \perp \xi$. However, the second case seems to be complex to analyze in practice. For this reason, we investigate for the case $V = \xi$.

By a simple computation of $(\mathcal{L}_g + 2S)$, we obtain

$$(\mathcal{L}_g)(X,Y) = 0. \quad \ldots (26)$$

$$h(\xi,\xi) = -2\lambda, \quad \ldots (27)$$

where $h(\xi,\xi) = (\mathcal{L}_g)(\xi,\xi) + 2S(\xi,\xi). \quad \ldots (28)$

Using (12) and (26) in above equation, we get

$$h(\xi,\xi) = 2\alpha^2(n-1). \quad \ldots (29)$$

Equating (27) and (29), we have

$$\lambda = -\alpha^2(n-1).$$

Since $\alpha$ is some non-zero scalar function, we have $\lambda \neq 0$, that is Ricci soliton in an $n$-dimensional $\alpha$-Kenmotsu manifold cannot be steady but is shrinking because $\lambda < 0$.

**Theorem 5.** If an $n$-dimensional $\alpha$-Kenmotsu manifold is $\eta$-Einstein then the Ricci solitons in $\alpha$-Kenmotsu manifold that is $(g,\xi,\lambda)$ where $\lambda = -\alpha^2(n-1)$ with varying scalar curvature cannot be steady but it is expanding.

**Proof.** The proof consists of three parts.

(i) We prove $\alpha$-Kenmotsu manifold is $\eta$-Einstein.

(ii) We prove the Ricci soliton in $\alpha$-Kenmotsu manifold is consisting of varying scalar curvature,

(iii) We find that the Ricci soliton in $\alpha$-Kenmotsu manifold is expanding.

First we prove that the $\alpha$-Kenmotsu manifold is $\eta$-Einstein: the metric $g$ is called $\eta$-Einstein if there exist two real functions $a$ and $b$ such that the Ricci tensor of $g$ is given by the general equation

$$S(X,Y) = ag(X,Y) + bn(X)\eta(Y). \quad \ldots (30)$$

Let $e_i (i = 1, 2, ..., n)$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i$ in (30) and taking summation over $i$, we get

$$r = a + b. \quad \ldots (31)$$

Again putting $X = Y = \xi$ in (30) then by use of (12), we have

$$a + b = -\alpha^2(n-1). \quad \ldots (32)$$

Then from (31) and (32), we have

$$a = \left(a^2 + \frac{\alpha}{n-1}\right), b = -\left(n a^2 + \frac{\alpha}{n-1}\right). \quad \ldots (33)$$

Substituting the value of $a$ and $b$ from (33) in (30), we have

$$S(X,Y) = \left(a^2 + \frac{\alpha}{n-1}\right)g(X,Y) - \left(n a^2 + \frac{\alpha}{n-1}\right)\eta(X)\eta(Y), \quad \ldots (34)$$

the above equation shows that $\alpha$-Kenmotsu manifold is $\eta$-Einstein manifold.

Now, we have to show that the scalar curvature $r$ is not a constant and it is varying
For an \( n \)-dimensional \( \alpha \)-Kenmotsu manifolds the symmetric parallel covariant tensor \( h(X, Y) \) of type \((0, 2)\) is given by
\[
h(X, Y) = (L_\alpha g)(X, Y) + 2S(X, Y). \quad \ldots(35)
\]
By using (25) and (34) in (35), we have
\[
h(X, Y) = 2\left\{a(a + 1) + \frac{r}{n-1}\right\} g(X, Y) - 2\left\{a(na + 1) + \frac{r}{n-1}\right\} \eta(X)\eta(Y). \quad \ldots(36)
\]
Differentiating (36) covariantly with respect to \( Z \) and using (14), we have
\[
(P_\alpha h)(X, Y) = 2\left\{(2a)(a + 1) + a(3a) + \frac{V_\alpha r}{n-1}\right\} g(X, Y)
- 2\left\{(2a)(na + 1) + a(3a) + \frac{V_\alpha r}{n-1}\right\} \eta(X)\eta(Y)
- 2\left\{a(na + 1) + \frac{r}{n-1}\right\} \eta(Z)\eta(X)
+ g(Z, Y)\eta(Z)\eta(Y). \quad \ldots(37)
\]
By substituting \( Z = \xi \) and \( X = Y \in (\text{Span})^2 \) in (37) and by using \( P_\alpha h = 0 \), we have
\[
P_\alpha r = (n-1)P_\alpha (a(\alpha + 1)). \quad \ldots(38)
\]
On integrating (38), we have
\[
r = (n-1)a(a + 1) + c, \quad \ldots(39)
\]
where \( c \) is some integral constant. Thus from (39), we have \( r \) is a varying scalar curvature.

Finally, we have to check the nature of the soliton that is Ricci soliton in \( \alpha \)-Kenmotsu manifold:

From (1), we have \( h(X, Y) - 2\lambda g(X, Y) \) then putting \( X = Y = \xi \), we have
\[
h(\xi, \xi) = -2\lambda. \quad \ldots(40)
\]
On putting \( X = Y = \xi \) in (36), we have
\[
h(\xi, \xi) = -2(n-1)\lambda. \quad \ldots(41)
\]
Equating (40) and (41), we have
\[
\lambda = (n-1)\lambda. \quad \ldots(42)
\]
This show that \( \lambda > 0 \), \( \forall \ n > 1 \) and hence Ricci soliton in an \( \alpha \)-Kenmotsu manifold is expending.

**Theorem 6.** If a Ricci soliton \((g, \xi, \lambda)\) where \( \lambda = 2a^2 \) of 3-dimensional \( \alpha \)-Kenmotsu manifold with varying scalar curvature cannot be steady but it is expending.

**Proof.** The proof consists of three parts.

(i) We prove that the Riemannian curvature tensor of 3-dimensional \( \alpha \)-Kenmotsu manifold is \( \eta \)-Einstein.

(ii) We prove that the Ricci soliton in 3-dimensional \( \alpha \)-Kenmotsu manifold is consisting of varying scalar curvature.

(iii) We prove that find that the Ricci soliton in a 3-dimensional \( \alpha \)-Kenmotsu manifold is expending.

The Riemannian curvature tensor of 3-dimensional \( \alpha \)-Kenmotsu manifold is given by
\[
R(X, Y)Z = g(Y, Z)QX - g(X, Z)Y + S(Y, Z)X - S(X, Z)Y - \frac{1}{\lambda} g(Y, Z)X - g(X, Z)Y. \quad \ldots(43)
\]
Putting \( Z = \xi \) in (42) and by using (8) and (11), we have
\[
\alpha^2(\eta(X)Y - \eta(Y)X) = \eta(Y)QX - \eta(X)QY - \left(2a^2 + \frac{\lambda}{2}\right)(\eta(Y)X - \eta(X)Y). \quad \ldots(44)
\]
Again putting \( Y = \xi \) in (43) and by using (2), (3) and (13), we get
\[
QX = \left(a^2 + \frac{\lambda}{2}\right)X - \left(3a^2 + \frac{\lambda}{2}\right)\eta(X)\xi. \quad \ldots(45)
\]
By taking an inner product with \( Y \) in (44), we have
\[
S(X, Y) = \left(a^2 + \frac{\lambda}{2}\right)g(X, Y) - \left(3a^2 + \frac{\lambda}{2}\right)\eta(X)\eta(Y). \quad \ldots(46)
\]
It shows that 3-dimensional \( \alpha \)-Kenmotsu manifold is \( \eta \)-Einstein manifold.

Now, we have to show that the scalar curvaturer \( r \) is not a constant that is \( r \) is varying. We have
\[
h(X, Y) = (L_\alpha g)(X, Y) + 2S(X, Y). \quad \ldots(47)
\]
By using (25) and (45) in (46), we have
\[
h(X, Y) = 2\{a(a + 1) + \frac{V_\alpha r}{2}\} g(X, Y) - 2\{a(3a + 1) + \frac{\lambda}{2}\} \eta(X)\eta(Y). \quad \ldots(48)
\]
Differentiating above equation with respect to \( Z \), we have
\[
(P_\alpha h)(X, Y) = 2\{2a(a + 1) + a(3a) + \frac{V_\alpha r}{2}\} g(X, Y)
- 2\{2a(3a + 1) + a(3a) + \frac{\lambda}{2}\} \eta(X)\eta(Y)
- 2\{a(3a + 1) + \frac{\lambda}{2}\} \{P_\alpha h\}(X)\eta(Y)
+ \eta(X)\{P_\alpha h\}(Y). \quad \ldots(49)
\]
By substituting \( Z = \xi \) and \( X = Y \in (\text{Span})^2 \) in (48) and by using \( P_\alpha h = 0 \), we have
\[
P_\alpha r = -2V_\alpha (a(\alpha + 1)). \quad \ldots(50)
\]
On integrating (49), we have
\[
r = -2a(\alpha + 1) + c, \quad \ldots(51)
\]
where \( c \) is some integral constant. Thus from (50), we have \( r \) is a varying scalar curvature.

Finally we have to check the nature of the Ricci soliton \((g, \xi, \lambda)\) in 3-dimensional \( \alpha \)-Kenmotsu manifold.

From (1), we have \( h(X, Y) = 2\lambda g(X, Y) \) then putting \( X = Y = \xi \), we have
\[
h(\xi, \xi) = -2\lambda. \quad \ldots(52)
\]
Equating (51) and (52), we have
\[
h(\xi, \xi) = -4\lambda. \quad \ldots(53)
\]
Equating (51) and (52), we have

\[ \lambda = 2a^2. \]

This shows that \( \lambda > 0 \) and hence Ricci soliton in an \( \alpha \)-Kenmotsu manifold is expanding.

References