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ALTERNATIVE ANALYSIS: THE PRIME NUMBERS THEORY AND AN EXTENSION OF THE REAL NUMBERS SET

АЛЬТЕРНАТИВНЫЙ АНАЛИЗ: ТЕОРИЯ ПРОСТЫХ ЧИСЕЛ И РАСШИРЕНИЕ МНОЖЕСТВА ДЕЙСТВИТЕЛЬНЫХ ЧИСЕЛ

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Abstract. Here we consider the theory of prime numbers at a new methodology. The theory of prime numbers is one of the most ancient mathematical branches. We found an estimate of the all prime numbers sum using the notions of infinite lager numbers and infinitely small numbers, farther we estimated the value of the maximal prime number. We proved that Hardy–Littlewood Hypothesis has the positive decision too. The infinite small numbers define a new methodology of the well–known function o(x) application. We use the sets of the theory of prime numbers and infinitely small numbers with a linear function $h(x) = \kappa x$ to formulate the alternative extension of the real numbers set.

Аннотация. В статье мы рассматриваем в новой методологии теорию простых чисел, которая является одной из древнейших областей математики. Используя понятия бесконечно больших и бесконечно малых чисел, мы получили оценку количества всех простых чисел, далее мы нашли оценку значения наибольшего простого числа. Мы также доказали, что гипотеза Харди–Литлвуда имеет положительное решение. Бесконечно малые числа определяют новую методологию применения хорошо известной функции o(x). Мы используем бесконечно большие и бесконечно малые числа и линейную функцию $h(x) = \kappa x$ для формулирования альтернативного расширения множества действительных чисел.

Keywords: First Euclidian theorem, the prime numbers, infinity large number, Hardy–Littlewood's Hypothesis, the existence of maximal prime number, Mersenne's prime numbers, the extension of the real numbers set.

Ключевые слова: Первая теорема Евклида, простые числа, бесконечно большие числа, гипотеза Харди–Литлвуда, существование наибольшего простого числа, простые числа Мерсенне, расширение множества действительных чисел.

1. The Main theorem of Arithmetic and the infinity of all prime numbers set

Greek mathematician Euclid, when he was as Professor of Alexandria University (roughly 300 BC), known and used the *Main theorem of Arithmetic* (Theorem 1) which was written with modern wording in [1, Th. 1.1.1]:

Theorem 1. There exists for every integer positive number *n* the decomposition on the product of the prime number–multipliers $p_i^{a_i}$ degrees:

$$n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}.$$

At the first, we formulate the *First Euclidian theorem* **(**Theorem 2) in new recent wording.

Theorem 2. The quantity of all prime numbers is not bounded with any positive number.

The proof of this basis Theorem is very simple [1, Th. 1.1.2]: Let the p be the last maximal prime number by our supposition, then let $n \triangleq 2 \cdot 3 \cdot 5 \cdot \dots \cdot p + 1$. So the *n* is a new prime number. Thus our assumption of a finiteness of prime numbers set contradicts to First Euclidian theorem. This proof delights all scientific world more than two with half thousand years. However, the truth was hided into an indefiniteness of the infinity notion. The authors of [1] write that starting from the most elementary and ancient ideas associated with a set of prime numbers, everybody can quickly reach the front edge of the modern scientific research [1, 1.1.3].

2. The sum of all prime numbers and Hardy–Littlewood's Hypothesis

Let $\boldsymbol{P}$ be [1, 1.1.3] the set of all prime numbers p_k, $k \in \pi \subset N$. Let farther $\boldsymbol{P}(x) \triangleq \{p_k : p_k \le x > 1\}$. Now let $\pi(x) \triangleq |\boldsymbol{P}(x)|$, then $\lim_{x\to\infty} \pi(x) = |\boldsymbol{P}|$, what is generally accepted. That is obviously that the graph $y = \pi(x)$ of function $\pi(x)$ has a consecutive form and the function $\pi(x)$ is a step–function with for all k $\pi(p_{k+1}) - \pi(p_k) = 1$. Let $g(x)$ be a differentiable function which has following complementary properties $g(p_k) = \pi(p_k)$. As it is well known [2, chap. 7.1] there exists any subset $\boldsymbol{AC} \triangleq \{(\boldsymbol{a})\}$ of the set of Cauchy sequences everyone of them, at the first, does not limited with any finite number and, secondly, holds the limit condition

$$lim(a_{n+1} - a_n) = 0. \qquad (1)$$

Every Cauchy sequences $(\boldsymbol{a}) \in \boldsymbol{AC}$ converges by virtue of (1) to some corresponding *infinite large number* (*ILN*($\boldsymbol{a}$)). Now let the set $\{ILN\} \triangleq \boldsymbol{\Omega}$. In the same place [2, Example 7.2.2] it was shown that

$$lim_{n\to\infty}\pi(x) = \pi(\infty) \triangleq \Omega_\pi \in \boldsymbol{\Omega}. \qquad (2)$$

The number Ω_π is the corresponding *ILN* (π) and defines the $|\boldsymbol{P}|$. It is obvious $\pi(\infty) = g(\infty) = \Omega_\pi$ too. At the second hand we can investigate the properties of differentiable convex function*s* $f: R_+ \to R$. The diagram *Gr*(*f*) of every that function f lies under its tangent *T*(*f*) in any point $M(x, f(x))$. Right now we support that the derivative $f'$of the f is a monotone nonnegative function with $f'(\infty) = 0$. That means so $\alpha \triangleq \lim_{x\to\infty} d(Gr(f), T(f))$=0 by virtue of the $d(A,B) \triangleq |A - B|$. In the others word, if $T(x, Y(x)) \in T(f), M(x,y) \in Gr(f)$ we have $Y(x) - y \ge 0$, thus $\alpha \triangleq \lim_{x\to\infty}(Y(x) - y)$=0. At last we say that the tangent *T*(*f*) will be at $x \to \infty$ as an asymptote to diagram *Gr*(*f*) of the function f. In this case we can say: The differentiable convex functions $f: R_+ \to R$ at $x \to \infty$ and with $f'(\infty) = 0$ defines some infinite large number *ILN*(*f*) as in [3, 7.2]. Now we write $f(\infty) \triangleq \Omega(f)$. Let, for example, the function $f(x)$ be the$\ln(x)$, then $f(\infty) = \Omega(f) \triangleq \Omega_e$. More we can show that there exists such finite number κ, $0 < \kappa \ll 1$, that the diagram *Gr*(*h*) of lineal function $h(x) = \kappa x$ and diagram *Gr*(*f*) of the function $f(x) = \ln(x)$ have unique common point $L(x_0, y_0)$ with $y_0 < \Omega_e$, Ω_e<*h* (∞) =∞. About this case we write either

$$\tilde{\infty} \triangleq \infty \backslash (\boldsymbol{\Omega} \cup \boldsymbol{R}) \text{ or } \tilde{\infty} \cup (\boldsymbol{\Omega} \cup \boldsymbol{R}) \triangleq \infty .$$

Thus $\tilde{\infty} \cap \boldsymbol{\Omega}=\emptyset$. Following to G. W. F. Hegel, the set $\tilde{\infty}$ is said to be "foolish infinity".
Farther we shall use an almost obvious statement (see Theorem 7.2.1 in [3])

Theorem 3. The unbounded differentiable in $\pm\infty$ *function* $f: R \to R$ *converges to corresponding ILN* $\Omega(f)$ *if and only if* the *function f has* $f'(\infty) = 0$.

Now using both the equalities (1), (2) and Theorem 3 we can prove that the defined above function $g(x)$ is a convex function at $x \to \infty$ and has at least asymptotic character. Hence the ***function*** $\boldsymbol{\pi(x)}$ ***has the convex character too*** at $x \to \infty$. Therefore, we proved that *Hardy–Littlewood's Hypothesis* [1, 1.2.4] *has the positive decision.*

3. The infinite small numbers and function $\boldsymbol{o(x)}$

Let the convex function $f: R \to R$, as and above, be the function which tends to $\Omega(f)$ at $x \to \infty$. Then the function $\omega(x) \triangleq 1/f(x)$ at $x \to \infty$ tends to zero and we shall write by a definition

$$\omega \triangleq \alpha(f) \triangleq \lim_{x\to\infty} (f(x))^{-1}. \quad (3)$$

However, we have $|\alpha(f)| > 0$ in (3) by virtue of $(\infty) = \Omega(e) < \infty$. Now the $\alpha(f)$ is said to be *infinite small number* (*ISN*) which is *defined by the function* $\omega(x)$. Yet let the set $\{ISN\} \triangleq \boldsymbol{\omega}$. It is obvious, that $sign(\Omega(f))= sign(\omega(f))$ and $\omega'(\infty) = 0$. The limiting condition (3) distinguishes between the set of all *infinite small numbers* and a set of all *infinite small in limit functions* (*ISLF*) $h(x)$, which holds $\lim_{x\to a} h(x) = 0$ [4, cap. 1.1, it. 24]. The infinite large and infinite small numbers are the *new essences* in the set of *Real Numbers* in contrast to the infinite small and infinite large in limit variables which are the variables of specific characters only. However, we know that there exists such *ISLF* $\eta(x)$ at $x \to b$ which holds

$$lim_{x\to b}\, \eta'(x) = \eta'(b) = 0. \quad (4)$$

By virtue of (4) we have $\lim_{x\to b} \eta(x) = \eta(b) \triangleq \omega(\eta)$ with $|\omega(\eta)|>0$. For example, let the function $\eta(x)$ be $\eta(x) \triangleq 1/lnx\,, b \triangleq \infty$. Now we have $\eta(b) = \alpha(\eta)$ which is an *ISN* and $\alpha(\eta) = 1/\Omega(e) > 0$. The use of well–known function $ISLF = o(x)$ ([1, cap.1.1.4], [4, cap.2.3, it.60]) simplifies the proofs with the infinitesimals. In some cases, the either factorization or decomposition into sum of *ISLF*, i. e. $f(x) = h(x) \cdot \eta(x)$ and, corresponding, $f(x) = g(x) + \lambda(x)$, there we have $\eta(\infty) \in \boldsymbol{\omega}$ and $\lambda(\infty) \in \boldsymbol{\omega}$**,** inserts much more definiteness of arguments by virtue of $\lambda(\infty) \cdot \eta(\infty) \neq 0$, as it will be show in following item 4. For example, the authors of [1] Richard Crandall and Carl Pomerance denoted the $\pi(x)/x$ [1. 1.1.4] by $A(x)/x$ and named $\lim_{x\to\infty} A(x)/x \triangleq d$ as an asymptotic density of the set A. Farther, they asserted that the set A has the zero asymptotic density by virtue of the asymptotic equality $\pi(x) = o(x)$. But, by virtue of [3, Example 7.2.2] we have $\pi(x)/x = 1/lnx + o(1/lnx) > 1/lnx$ and by virtue of (4) $\lim 1/lnx = \alpha(\eta) = 1/\Omega(e) \in \boldsymbol{\omega}$, and $\alpha(\eta)>0$, as it was be shown above in this item again. Thus we have following

Conclusion: $lim_{x\to\infty} (\pi(x)/x) = lim_{x\to\infty}(1/lnx) \triangleq \alpha(z)>0$ i.e. $\alpha(z) \in \boldsymbol{\omega}$**.**

This result contradicts to $\pi(x) = o(x)$.

4. The maximal prime number, Mersenne's primes and some hypothesis

Further, let $p(x) \triangleq \max_{\boldsymbol{P}(x)}\{p_k\}$, then the function $z(p(x)) \triangleq \pi(x)/p(x)$, $x \in R$, determines the relative density of the Prime numbers distribution at every point $p(x)$. It is obviously by [4, 3.3.1] and (2), (3)

$$z(p(x)) \geq z(x) = \pi(x)/x = 1/lnx + o(1/lnx) > 1/lnx > 0 \quad (5)$$

Now following to Table 1.1 in the [1, 1.1.5] we have following new Table of some values of function z(x):

Table.

SOME VALUES OF FUNCTION Z(X)

x	10^2	10^3	10^4	10^6	10^8	10^{12}	10^{16}	10^{17}	10^{18}	10^{19}	10^{20}	10^{21}	10^{22}	4×10^{22}
10^4z (x)	2500	1681	1229	785	576	376	279	262	247	234	222	211	203	196

Now we have from Table that the function $z(x)$ is monotone one and the velocity of function $z(x)$ decrease diminishes at the x growth with $0{,}25 \geq z(x) \geq 0{,}005$. By our Table we write *a hypothetical equality* $z(x) = 10^{-3}a(x)$ at $\lim_{x\to\infty} a(x) = a$, at $a > 1$. Thus we have following approximate equality

$$p(x) \cong 10^3 a(x)^{-1}\,\pi(x). \tag{6}$$

Let $\lim_{k\in\pi} p_k \triangleq p_\Omega$ be the maximal prime number. It is obviously that we can write either $\exists\{\Omega(f_1), \Omega(f_2)\}\colon \Omega(f_1) < p_{max} < \Omega(f_2)$, or $p_{max} \in \mathbf{\Omega}$. At last we have new *Hypothesis — the estimate of maximal prime number* by virtue of (2), (6) in following form

$$p_{max} = 10^3 a^{-1}\cdot \Omega_\pi,\ at > 1. \tag{7}$$

In general case the equality (7) contains three variables: p_{max}, Ω_π, a. If we know the properties of function $z(x) = 10^{-3}a(x)$ more exactly we can to know the values of the function $\pi(x)$ at corresponding prime number. So we have following estimate from (6)

$$\pi(x) \cong p(x)10^{-3}a(x) \tag{8}$$

In particular, it is well–known the prime numbers of Maren Mersenne (1588–1648) have the following form: $M_q \triangleq 2^q - 1$ [1, 1.3.1]. The largest (2005) of all Mersenne's prime numbers is equal the $p_k = 2^{25964951} - 1$. What the index k has this p_k? By the definitions of $\pi(x)$ and $p(x)$ we have $\pi(x)_{x=p_k} \triangleq k$. Let by *our new hypothesis*

$$a(p_k) \triangleq 2\cdot 10.$$

Now we have from (8) $k \cong (2^{25964951} - 1)\cdot 2\cdot 10^{-2}$=$(2^{25964952})\cdot 10^{-2}$. Thus we have

$$|\boldsymbol{P}|_{max(2016)} \cong (2^{25964952})\cdot 10^{-2}.$$

5. An extension of the real numbers set

As it is accepted in an analysis $\{\boldsymbol{R}, \pm\infty\} \triangleq \overline{\boldsymbol{R}}$. Let, as in the item 2, $\infty\backslash(\mathbf{\Omega} \cup \boldsymbol{R}) \triangleq \tilde{\infty}$.

The contents of items 1–4 allow to make the following alternative extension of the real numbers set $\boldsymbol{R}$ into $\widetilde{\boldsymbol{R}}$. At the first, we will consider a linear function $h\colon \overline{\boldsymbol{R}} \to \overline{\boldsymbol{R}}$, determined by a formula $h(x) = \kappa x, -1 \leq k \leq 1$. It is obvious in analysis that

$$\forall x\ k = 0 \Rightarrow h(x) = 0\ and\ \forall k, 0 < k \leq 1, x = \infty \Rightarrow h(\infty) = \infty.$$

Further we will designate a limit value $h(\infty)$ of function h by a symbol $\infty_k, 0 < k \leq 1$. Let $\langle\infty\rangle \triangleq \max_\infty\{x, x \in \tilde{\infty}\}$, then $\langle\infty\rangle = \infty_{k|k=1} = \infty_1$. Let $k \triangleq \alpha, 0 < \alpha \in \boldsymbol{\omega}$ then we write

$$h(\infty_1) = \alpha\cdot\infty_1 \triangleq \Omega_\alpha \in \mathbf{\Omega}. \tag{9}$$

Thus linear function $h\colon \bar{R} \to \bar{R}$ defines the mapping

$$\boldsymbol{\omega} \ \to \boldsymbol{\Omega}(h) \subseteq \boldsymbol{\Omega}. \qquad (10)$$

On the others hand the every $ILN(f)$ defines by (3) any corresponding $\omega \triangleq \alpha(f)$. By virtue of (3), (9) the right part of (10) cannot has the strong inclusion.

Thus, we have allocated from the volume of concept $\widetilde{\infty}$ the maximal elements $\pm\infty_1$ and the set $\{\infty_k, 0 < |k| \leq 1, k \notin \omega\}$ of infinite elements as "foolish infinity":

$$\boldsymbol{R} \cup \boldsymbol{\Omega} \cup \{\{\pm\infty_k, 0 < k < 1, k \notin \omega\} \cup \{\pm \infty_1\} \triangleq \widetilde{\boldsymbol{R}}.$$

From the logical point of view it is obvious that $\widetilde{\boldsymbol{R}} = \overline{\boldsymbol{R}}$. At the third, our future readers will be to have some questions. Here we give our answers to two of them.

1) The sets (0,1), and $\widetilde{\boldsymbol{R}} \backslash (\boldsymbol{R} \cup \boldsymbol{\Omega})$ are not bijective sets, by virtue of the Euclid Axiom 8-th.

2) Euclid has proved the inequality $\cap\, \boldsymbol{\Omega} \neq \emptyset$. Our answer is, of course: "It is very possible, perhaps".

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