PREŠIĆ TYPE FIXED POINT THEOREM
FOR SIX MAPS IN $D^*$- METRIC SPACES

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Abstract. In this paper, we obtain a Prešić type fixed point theorem for three pairs of jointly $3k$-weakly compatible maps in $D^*$-metric spaces. We also present an example to illustrate our main theorem. We also obtain four corollaries for four maps, three maps, two maps and a single map. We also give some probable modifications of Theorems of [5, 12, 13] in $G$-metric spaces.

1. Introduction and Preliminaries

In 1922, Banach [6] proved a theorem which is known as Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Later many authors have extended, generalized and improved Banach’s fixed point theorem in different ways. In 1992, Dhage [1] introduced generalized metric space or $D$-metric space and proved several results.

Naidu et al [9], [10], [11] observed that almost all fixed point theorems in $D$-metric spaces are not valid or of doubtful validity and modified some fixed point theorems in $D$-metric spaces. As a probable modification of $D$-metric spaces, Sedghi et al. [8] introduced $D^*$-metric spaces and Mustafa et al. [14] introduced $G$-metric spaces.

On the other hand, amongst the various generalizations of Banach contraction principle, Prešić [7] gave a contractive condition and proved a Banach type fixed point theorem which was useful to solve certain difference equations. Throughout this paper $\mathbb{N}$ denotes the set of all positive integers.

Actually Prešić [7] proved the following.

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THEOREM 1.1. ([7]) Let \((X, d)\) be a complete metric space, \(k\) a positive integer and \(f : X^k \to X\). Suppose that
\[
d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \leq \sum_{i=1}^{k} q_i d(x_i, x_{i+1})
\]
holds for all \(x_1, x_2, \ldots, x_k, x_{k+1} \in X\), where \(q_i > 0\) and \(\sum_{i=1}^{k} q_i \in [0, 1)\). Then \(f\) has a unique fixed point \(x^*\). Moreover, for any arbitrary points \(x_1, x_2, \ldots, x_{k+1}\) in \(X\), the sequence \(\{x_n\}\) defined by \(x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1})\), for all \(n \in \mathbb{N}\) converges to \(x^*\).

Later Ćirić and Prešić [4] generalized the above theorem as follows.

THEOREM 1.2 ([4]). Let \((X, d)\) be a complete metric space, \(k\) a positive integer and \(f : X^k \to X\). Suppose that
\[
d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \leq \lambda \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}
\]
holds for all \(x_1, x_2, \ldots, x_k, x_{k+1} \in X\), where \(\lambda \in [0, 1)\). Then \(f\) has a fixed point \(x^* \in X\). Moreover, for any arbitrary points \(x_1, x_2, \ldots, x_{k+1}\) in \(X\), the sequence \(\{x_n\}\) defined by \(x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1})\), for all \(n \in \mathbb{N}\) converges to \(x^*\). Moreover, if \(d(f(u, v, \ldots, v), f(v, v, \ldots, v)) < d(u, v)\) holds for all \(u, v \in X\) with \(u \neq v\), then \(x^*\) is the unique fixed point of \(f\).

Recently Rao et al. [2], [3] obtained some Prešić type theorems for two and three maps in metric spaces. Now we give the following definition of [2], [3].

DEFINITION 1.1. Let \(X\) be a nonempty set and \(T : X^{2k} \to X\) and \(f : X \to X\). The pair \((f, T)\) is said to be \(2k\)-weakly compatible if \(f(T(x, x, \ldots, x)) = T(fx, fx, \ldots, fx, fx)\) whenever \(x \in X\) such that \(fx = T(x, x, \ldots, x)\).

Using this definition, Rao et al. [2] proved the following

THEOREM 1.3 ([2]). Let \((X, d)\) be a metric space, \(k\) a positive integer and \(S, T : X^{2k} \to X, f : X \to X\) be mappings satisfying
\[
\begin{align*}
(1.3.1) & \quad d\left(\frac{S(x_1, x_2, \ldots, x_{2k})}{T(x_2, x_3, \ldots, x_{2k+1})}\right) \leq \lambda \max\{d(fx_i, fx_{i+1}) : 1 \leq i \leq 2k\} \\
& \text{for all } x_1, x_2, \ldots, x_{2k}, x_{2k+1} \in X, \\
(1.3.2) & \quad d\left(\frac{T(y_1, y_2, \ldots, y_{2k})}{S(y_2, y_3, \ldots, y_{2k+1})}\right) \leq \lambda \max\{d(fy_i, fy_{i+1}) : 1 \leq i \leq 2k\} \\
& \text{for all } y_1, y_2, \ldots, y_{2k}, y_{2k+1} \in X, \text{ where } 0 < \lambda < 1
\end{align*}
\]
(1.3.3) \(d(S(u, \ldots, u), T(v, \ldots, v)) < d(fu, fv)\), for all \(u, v \in X\) with \(u \neq v\)
(1.3.4) Suppose that \(f(X)\) is complete and either \((f, S)\) or \((f, T)\) is a \(2k\)-weakly compatible pair.

Then there exists a unique point \(p \in X\) such that \(fp = p = S(p, \ldots, p) = T(p, \ldots, p)\).

In this paper we obtain a Prešić type common fixed point theorem for six mappings in \(D^*\)-metric spaces and present an example to illustrate our main theorem.
Now we give some known definitions and lemmas which are useful for further discussion.

**Definition 1.2 ([14]).** Let $X$ be a non-empty set and let $G : X^3 \to \mathbb{R}^+$ be a function satisfying the following properties:

\[(G_1)\] $G(x, y, z) = 0$ if and only if $x = y = z$,
\[(G_2)\] $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
\[(G_3)\] $G(x, y, z) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
\[(G_4)\] $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, symmetry in all three variables,
\[(G_5)\] $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function $G$ is called a generalized metric or a $G$-metric on $X$ and the pair $(X, G)$ is called a $G$-metric space.

Recently Dhasmana [5] and Gairola et al. [12], [13] proved Prešić type fixed and common fixed point theorems in $G$-metric spaces. They are the following theorems

**Theorem 1.4 (Theorem 2.1, [5]).** Let $(X, G)$ be a complete $G$-metric space, $k$ a positive integer and $T : X^k \to X$ a mapping satisfying the following contractive type condition

\[
(1.4.1) \quad G \left( \begin{array}{c} T(x_1, x_2, \ldots, x_k), \\ T(x_2, x_3, \ldots, x_{k+1}), \\ T(x_3, x_4, \ldots, x_{k+2}) \end{array} \right) \leq \lambda \max \{G(x_i, x_{i+1}, x_{i+2}) : 1 \leq i \leq k\}
\]

where $\lambda \in (0, 1)$ is constant and $x_1, x_2, \ldots, x_{k+2}$ are arbitrary elements in $X$. Then there exists a point $x$ in $X$ such that $x = T(x, x, \ldots, x)$.

Moreover, if $x_1, x_2, \ldots, x_k$ are arbitrary points in $X$ and for $n \in \mathcal{N}$, $x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$ then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \ldots, \lim x_n)$.

If, in addition we suppose that on diagonal $\Delta \subset X^k$,

\[
G \left( \begin{array}{c} T(u, u, \ldots, u), \\ T(v, v, \ldots, v), \\ T(w, w, \ldots, w) \end{array} \right) < G(u, v, w)
\]

holds for all $u, v, w \in X$ with $u \neq v \neq w$, then $x$ is unique point in $X$ with $T(x, x, \ldots, x) = x$.

**Theorem 1.5 (Theorem 3.1, [12]).** Let $(X, G)$ be a $G$-metric space, $k$ a positive integer and $T : X^k \to X$, $f : X \to X$ be mappings satisfying the following conditions

\[
(1.5.1) \quad T(X^k) \subseteq f(X), \\
(1.5.2) \quad G \left( \begin{array}{c} T(x_1, x_2, \ldots, x_k), \\ T(x_2, x_3, \ldots, x_{k+1}), \\ T(x_3, x_4, \ldots, x_{k+2}) \end{array} \right) \leq \lambda \max \{G(fx_i, fx_{i+1}, fx_{i+2}) : 1 \leq i \leq k\}
\]

for all $x_1, x_2, \ldots, x_{k+2} \in X$, where $0 \leq \lambda < 1$;
\[
(1.5.3) \quad d(T(u, \ldots, u), T(v, \ldots, v), T(w, \ldots, w)) < G(fu, fv, fw),
\]
for all \( u, v, w \in X \) with \( u \neq v \neq w \),

(1.5.4) \( f(X) \) is \( G \)-complete and if the pair \((f, T)\) is coincidentally commuting.

Then there exist a unique point \( p \in X \) such that \( fp = p = T(p, p, \ldots, p) \).

**Theorem 1.6 (Theorem 3.1, [13]).** Let \((X, G)\) be a \( G \)-metric space, \( k \) a positive integer and \( S, T, R : X^k \to X \), \( f : X \to X \) be mappings satisfying the following conditions

(1.6.1) \( S(X^k) \cup T(X^k) \cup R(X^k) \subseteq f(X) \),

(1.6.2) \( G \left( \begin{array}{c} S(x_1, x_2, \ldots, x_k) \\ T(x_2, x_3, \ldots, x_{k+1}) \\ R(x_3, x_4, \ldots, x_{k+2}) \end{array} \right) \leq \lambda \max \{ G(fx_i, fx_{i+1}, fx_{i+2}), 1 \leq i \leq k \} \)

for all \( x_1, x_2, \ldots, x_{k+2} \in X \),

(1.6.3) \( G \left( \begin{array}{c} T(y_1, y_2, \ldots, y_k) \\ R(y_2, y_3, \ldots, y_{k+1}) \\ S(y_3, y_4, \ldots, y_{k+2}) \end{array} \right) \leq \lambda \max \{ G(fy_i, fy_{i+1}, fy_{i+2}), 1 \leq i \leq k \} \)

for all \( y_1, y_2, \ldots, y_{k+2} \in X \),

(1.6.4) \( G \left( \begin{array}{c} R(z_1, z_2, \ldots, z_k) \\ S(z_2, z_3, \ldots, z_{k+1}) \\ T(z_3, z_4, \ldots, z_{k+2}) \end{array} \right) \leq \lambda \max \{ G(fz_i, fz_{i+1}, fz_{i+2}), 1 \leq i \leq k \} \)

for all \( z_1, z_2, \ldots, z_{k+2} \in X \),

(1.6.5) \( d(S(u, \ldots, u), T(v, \ldots, v), R(w, \ldots, w)) < G(fu, fv, fw) \)

for all \( u, v, w \in X \) with \( u \neq v \neq w \).

Suppose that \( f(X) \) is complete and one of \((f, S), (f, T)\) or \((f, R)\) is coincidentally commuting pair. Then there exists a unique point \( p \in X \) such that \( fp = p = S(p, p, \ldots, p) = T(p, p, \ldots, p) = R(p, p, \ldots, p) \).

We observed that in these theorems the authors [5, 12, 13] wrongly used the condition \((G_2)\) in proving Cauchy sequences.

We also observed the following:

(i) In Theorem 2.1 of [5], the condition (2.1.2) is also wrongly used. In Page 13 line 21 from below \( y \neq x \neq z, \ldots \), which is a contradiction. From this we can not conclude \( y = x = z \) only. There are some more possibilities namely \( x = y \) or \( y = z \) or \( x = z \).

(ii) In Theorem 3.1 of [12], the condition (3.3) is wrongly used two times. In Page 199, line 5 from above and line 10 from above.

(iii) In Theorem 3.1 of [13], the condition (5) is wrongly used two times. In line 3 from below of Page 403 and line 4 from above of Page 404.

**Definition 1.3 ([8]).** Let \( X \) be a non-empty set and \( D^* : X^3 \to \mathcal{R}^+ \) be a function satisfying:

(1.3.1) \( D^*(x, y, z) = 0 \) if and only if \( x = y = z \),

(1.3.2) \( D^*(x, y, z) = D^*(p\{x, y, z\}) \), where \( p \) is a permutation function,
(1.3.3): \( D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z) \)

Then the function \( D^* \) is called a \( D^* \)-metric and the pair \((X, D^*)\) is called a \( D^* \)-metric space.

**Remark 1.1** ([8]). In a \( D^* \)-metric space, we have \( D^*(x, x, y) = D^*(x, y, y) \) for all \( x, y \in X \).

**Definition 1.4** ([8]). Let \((X, D^*)\) be a \( D^* \)-metric space. For \( r > 0 \), define

\[
B_{D^*}(x, r) = \{ y \in X : D^*(x, y, y) < r \}
\]

**Definition 1.5** ([8]). Let \((X, D^*)\) be a \( D^* \)-metric space.

(i) If for every \( x \in A \subset X \), there exists \( r > 0 \) such that \( B_{D^*}(x, r) \subset A \), then \( A \) is called an open subset of \( X \).

(ii) A sequence \( \{x_n\} \) in \( X \) is said to be convergent to \( x \in X \) if and only if \( \lim_{n \to \infty} D^*(x_n, x, x) = 0 \).

(iii) A sequence \( \{x_n\} \) in \( X \) is called a Cauchy sequence if and only if

\[
\lim_{n, m \to \infty} D^*(x_n, x_m, x_m) = 0.
\]

(iv) \((X, D^*)\) is said to be complete if every Cauchy sequence is convergent in \( X \).

**Lemma 1.1** ([8]). Let \((X, D^*)\) be a \( D^* \)-metric space. Then \( D^* \) is continuous in all its three variables.

**Lemma 1.2** ([8]). If a sequence \( \{x_n\} \) in \((X, D^*)\) converges to \( x \in X \) then \( x \) is unique. Also \( \{x_n\} \) is a Cauchy sequence in \( X \).

**Definition 1.6** ([8]). Let \((X, D^*)\) be a \( D^* \)-metric space. Then \( D^* \) is called of first type if \( D^*(x, x, y) \leq D^*(x, y, z) \) for all \( x, y, z \in X \).

Now we are ready to prove our main theorem.

**2. Main Results**

**Theorem 2.1.** Let \((X, D^*)\) be a complete \( D^* \)-metric space where \( D^* \) is of first type, \( k \) a positive integer and \( S, T, R : X^{3k} \to X \) and \( f, g, h : X \to X \) be mappings satisfying

\[
\begin{align*}
(2.1.1) \quad S(X^{3k}) &\subseteq g(X), \; T(X^{3k}) \subseteq h(X), \; R(X^{3k}) \subseteq f(X), \\
(2.1.2) \quad D^*(S(x_1, x_2, \ldots, x_{3k}), T(y_1, y_2, \ldots, y_{3k}), R(z_1, z_2, \ldots, z_{3k})) \\
&\leq \lambda \max \left\{ D^*(g_{x_1}, h_{y_1}, f_{z_1}), D^*(h_{x_2}, f_{y_2}, g_{z_2}), D^*(f_{x_3}, g_{y_3}, h_{z_3}), \ldots, D^*(g_{x_{3k-2}}, h_{y_{3k-2}}, f_{z_{3k-2}}), D^*(h_{x_{3k-1}}, f_{y_{3k-1}}, g_{z_{3k-1}}), D^*(f_{x_{3k}}, g_{y_{3k}}, h_{z_{3k}}) \right\}
\end{align*}
\]
for all \( x_1, x_2, \ldots, x_{3k}, y_1, y_2, \ldots, y_{3k}, z_1, z_2, \ldots, z_{3k} \in X \) and \( 0 < \lambda < 1 \),

(2.1.3) The pairs \((S, f), (T, g)\) and \((R, h)\) are jointly \(3k\)-weakly compatible pairs.

(2.1.4) Suppose \( z = f u = g u = h u \) for some \( u \in X \) whenever there exists a sequence \( \{y_{3k+n}\}_{n=1}^{\infty} \) in \( X \) such that \( y_{3k+n} \to z \in X \) as \( n \to \infty \).

Then \( z \) is the unique point in \( X \) such that

\[
fz = gz = nz = s(z, z, \ldots, z) = T(z, z, \ldots, z) = R(z, z, \ldots, z).
\]

**Proof.** Suppose \( x_1, x_2, \ldots, x_{3k} \) are arbitrary points in \( X \). Define

\[
y_{3k+3n-2} = S(x_{3n-2}, x_{3n-1}, \ldots, x_{3k+3n-3}) = gx_{3k+3n-2},
\]

\[
y_{3k+3n-1} = T(x_{3n-1}, x_{3n}, \ldots, x_{3k+3n-2}) = hx_{3k+3n-1},
\]

\[
y_{3k+3n} = R(x_{3n}, x_{3n+1}, \ldots, x_{3k+3n-1}) = fx_{3k+3n} \text{ for } n = 1, 2, \ldots
\]

Let

\[
\alpha_{3n-2} = D^*(gx_{3n-2}, hx_{3n-1}, fx_{3n}),
\]

\[
\alpha_{3n-1} = D^*(hx_{3n-1}, fx_{3n}, gx_{3n+1}),
\]

\[
\alpha_{3n} = D^*(fx_{3n}, gx_{3n+1}, hx_{3n+2}), \quad n = 1, 2, \ldots
\]

and let \( \theta = \lambda \frac{1}{\lambda} \) and \( \mu = \max\{\frac{\alpha_{3n}}{\theta}, \frac{\alpha_{3n-1}}{\theta}, \frac{\alpha_{3n-2}}{\theta}\} \). Then \( \theta < 1 \) and by the selection of \( \mu \), we have

\[
\alpha_n \leq \mu \theta^n, \text{ for } n = 1, 2, \ldots, 3k.
\]

Consider

\[
\alpha_{3k+1} = D^*(gx_{3k+1}, hx_{3k+2}, fx_{3k+3})
\]

\[
= D^*(S(x_1, x_2, \ldots, x_{3k}), T(x_2, x_3, \ldots, x_{3k+1}), R(x_3, x_4, \ldots, x_{3k+2}))
\]

\[
\leq \lambda \max \left\{ D^*(fx_3, gx_4, hx_5), \ldots, D^*(gx_{3k-2}, hx_{3k-1}, fx_{3k}),
\right.
\]

\[
\left. D^*(hx_{3k-1}, fx_{3k}, gx_{3k+1}), D^*(fx_{3k}, gx_{3k+1}, hx_{3k+2}) \right\}
\]

\[
= \lambda \max \{ \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{3k-2}, \alpha_{3k-1}, \alpha_{3k} \}
\]

\[
\leq \lambda \max \{ \mu \theta, \mu \theta^2, \mu \theta^3, \ldots, \mu \theta^{3k-2}, \mu \theta^{3k-1}, \mu \theta^{3k} \}, \text{ from (2.1)}
\]

\[
= \lambda \mu \theta = \theta^{3k} \mu \theta
\]

\[
= \mu \theta^{3k+1}
\]

(2.3)

\[
\alpha_{3k+2} = D^*(hx_{3k+2}, fx_{3k+3}, gx_{3k+4}) = D^*(gx_{3k+4}, hx_{3k+2}, fx_{3k+3})
\]

\[
= D^*(S(x_4, x_5, \ldots, x_{3k+3}), T(x_5, x_6, \ldots, x_{3k+4}), R(x_6, x_7, \ldots, x_{3k+5}))
\]

\[
\leq \lambda \max \left\{ D^*(hx_5, fx_6, gx_7), \ldots, D^*(gx_{3k+4}, hx_{3k+2}, fx_{3k+3}),
\right.
\]

\[
\left. D^*(hx_{3k+2}, fx_{3k+3}, gx_{3k+4}), D^*(fx_{3k+3}, gx_{3k+4}, hx_{3k+5}) \right\}
\]

\[
= \lambda \max \{ \alpha_2, \alpha_3, \alpha_4, \alpha_5, \ldots, \alpha_{3k-1}, \alpha_{3k}, \alpha_{3k+1} \}
\]

\[
\leq \lambda \max \{ \mu \theta^2, \mu \theta^3, \mu \theta^4, \ldots, \mu \theta^{3k-2}, \mu \theta^{3k-1}, \mu \theta^{3k+1} \}, \text{ from (2.1), (2.2)}
\]

\[
= \lambda \mu \theta^2 = \theta^{3k} \mu \theta^2
\]

\[
= \mu \theta^{3k+2}
\]
Continuing in this way, we get
\[(2.4) \quad \alpha_{3k+3} = D^*(f_{3k+3}, gx_{3k+4}, hx_{3k+5}) = D^*(gx_{3k+4}, hx_{3k+5}, f_{3k+3})
\]
\[= D^*(S(x_{3k+3}, x_{3k+4}, \ldots, x_{3k+3}), T(x_{3k+4}, x_{3k+5}, \ldots, x_{3k+2}), R(x_{3k+5}, x_{3k+6}, \ldots, x_{3k+2}))
\]
\[\leq \lambda \max \left\{ \begin{array}{l}
D^*(gx_4, hx_5, f_{x_3}), D^*(hx_5, f_x, gx_4), \\
D^*(f_6, gx_7, hx_3), \ldots, D^*(gx_{3k+1}, hx_{3k+2}, f_{x_{3k+2}}), \\
D^*(hx_{3k+2}, f_{x_{3k+3}}, gx_{3k+1}), D^*(f_{x_{3k+3}}, gx_{3k+4}, hx_{3k+2})
\end{array} \right\}
\]
\[= \lambda \max \{ \alpha_3, \alpha_4, \alpha_5, \ldots, \alpha_{3k}, \alpha_{3k+1}, \alpha_{3k+2} \}
\]
\[\leq \lambda \max \{ \mu \theta^3, \mu \theta^4, \mu \theta^5, \ldots, \mu \theta^{3k}, \mu \theta^{3k+1}, \mu \theta^{3k+2} \}, \text{ from (2.1), (2.2), (2.3)}
\]
\[= \lambda \mu \theta^3 = \theta^{3k} \mu \theta^3
\]
\[= \mu \theta^{3k+3}.
\]
Continuing in this way, we get
\[(2.5) \quad \alpha_n \leq \mu \theta^n \text{ for all } n \in \mathbb{N}.
\]
Consider
\[D^*(y_{3k+3n+1}, y_{3k+3n+2}, y_{3k+3n+3})
\]
\[= D^*(gx_{3k+3n+1}, hx_{3k+3n+2}, f_{x_{3k+3n+3}})
\]
\[= D^*(S(x_{3k+1}, x_{3k+2}, \ldots, x_{3k+3}), T(x_{3k+2}, x_{3k+3}, \ldots, x_{3k+3}), R(x_{3k+3}, x_{3k+4}, \ldots, x_{3k+3}))
\]
\[\leq \lambda \max \left\{ D^*(gx_{3k+1}, hx_{3k+2}, f_{x_{3k+3}}), D^*(hx_{3k+2}, f_{x_{3k+3}}, gx_{3k+1}), D^*(f_{x_{3k+3}}, gx_{3k+4}, hx_{3k+2}) \right\}
\]
\[= \lambda \max \{ \alpha_{3n+1}, \alpha_{3n+2}, \alpha_{3n+3}, \ldots, \alpha_{3k+3n-2}, \alpha_{3k+3n-1}, \alpha_{3k+3n} \}
\]
\[\leq \lambda \max \{ \mu \theta^{3n+1}, \mu \theta^{3n+2}, \mu \theta^{3n+3}, \ldots, \mu \theta^{3k+3n-2}, \mu \theta^{3k+3n-1}, \mu \theta^{3k+3n} \}, \text{ from (2.5)}
\]
\[= \lambda \mu \theta^{3n+1} = \theta^{3k} \mu \theta^{3n+1}
\]
\[= \mu \theta^{3k+3n+1}
\]
Similarly, we have
\[D^*(y_{3k+3n+2}, y_{3k+3n+3}, y_{3k+3n+4}) \leq \mu \theta^{3k+3n+2}
\]
and
\[D^*(y_{3k+3n+3}, y_{3k+3n+4}, y_{3k+3n+5}) \leq \mu \theta^{3k+3n+3}
\]
Thus
\[(2.6) \quad D^*(y_{3k+n}, y_{3k+n+1}, y_{3k+n+2}) \leq \mu \theta^{3k+n}, \quad n = 1, 2, \ldots
\]
Since \( D^* \) is of first type, we have
\[(2.7) \quad D^*(y_{3k+n}, y_{3k+n}, y_{3k+n+1}) \leq D^*(y_{3k+n}, y_{3k+n+1}, y_{3k+n+2}) \leq \mu \theta^{3k+n}, \quad n = 1, 2, \ldots, \text{ from (2.6)}.
\]
Now for $m > n$, consider
\[
D^* (y_{3k+n}; y_{3k+n}, y_{3k+n+1}) + D^* (y_{3k+n+1}, y_{3k+n+1}, y_{3k+n+2}) + \cdots
\]
\[
+ D^* (y_{3k+m-1}, y_{3k+m-1}, y_{3k+m})
\]
\[
\leq \mu \theta^{3k+n} + \mu \theta^{3k+n+1} + \mu \theta^{3k+n+2} + \cdots + \mu \theta^{3k+m-1}
\]
\[
\leq \mu \theta^{3k} \frac{\theta^n}{1-\theta}, \text{ since } 0 < \theta < 1.
\]
Hence the sequence $\{y_{3k+n}\}$ is a Cauchy sequence in $X$. Since $X$ is complete, there exists $z \in X$ such that
\[
\lim_{n \to \infty} y_{3k+n} = z.
\]

By (2.1.4), there exists $u \in X$ such that
\[
z = fu = gu = hu
\]
Now consider
\[
D^* \left( \begin{array}{c}
S(u, u, \cdots, u), \\
T(x_{3n-1}, x_{3n}, \cdots, x_{3k+3n-2}), \\
R(x_{3n}, x_{3n+1}, \cdots, x_{3k+3n-1})
\end{array} \right)
\]
\[
\leq \lambda \max \left\{ D^* (gu, hx_{3n-1}, fx_{3n}), D^* (hu, fx_{3n}, gx_{3n+1}), \\
D^* (fu, gx_{3n+1}, hx_{3n+2}), \cdots, D^* (gu, hx_{3k+3n-4}, fx_{3k+3n-3}), \\
D^* (hu, fx_{3k+3n-3}, gx_{3k+3n-2}), D^* (fu, gx_{3k+3n-2}, hx_{3k+3n-1}) \right\}
\]
Letting $n \to \infty$ and using (2.8), (2.9) we get
\[
D^* (S(u, u, \cdots, u, u), fu, fu) \leq \lambda(0)
\]
which in turn yields that
\[
S(u, u, \cdots, u) = fu
\]
Similarly we can show that
\[
T(u, u, \cdots, u) = gu
\]
\[
R(u, u, \cdots, u) = hu
\]
Since the pairs $(S, f)$, $(T, g)$ and $(R, h)$ are jointly $3k$-weakly compatible and from (2.10), (2.11), (2.12), we have
\[
fz = f(fu) = f(S(u, u, \cdots, u)) = S(fu, fu, \cdots, fu) = S(z, z, \cdots, z)
\]
\[
gz = T(z, z, \cdots, z)
\]
and
\[
hz = R(z, z, \cdots, z)
\]
Now consider
\[
D^*(fz, z, z) = D^* \left( \begin{array}{c} S(z, z, \cdots, z), \\ T(u, u, \cdots, u), \\ R(u, u, \cdots, u) \end{array} \right), \quad \text{from (2.11), (2.12) and (2.13)}
\]
\[
\leq \lambda \max \left\{ D^*(gz, hu, fu), D^*(hz, fu, gu), \\ D^*(fz, gu, hu), \cdots, D^*(gz, hu, fu), \\ D^*(hz, fu, gu), D^*(fz, gu, hu) \right\}
\]
\[
= \lambda \max\{D^*(gz, z, z), D^*(hz, z, z), D^*(fz, z, z)\}.
\]

Similarly, we can show that
\[
D^*(z, gz, z) \leq \lambda \max\{D^*(gz, z, z), D^*(hz, z, z), D^*(fz, z, z)\},
\]
\[
D^*(z, z, hz) \leq \lambda \max\{D^*(gz, z, z), D^*(hz, z, z), D^*(fz, z, z)\}.
\]
Thus
\[
\max \left\{ D^*(fz, z, z), \\ D^*(gz, z, z), \\ D^*(hz, z, z) \right\} \leq \lambda \max \left\{ D^*(fz, z, z), \\ D^*(gz, z, z), \\ D^*(hz, z, z) \right\}
\]
which implies that \( fz = gz = hz = z \).

Now from (2.13), (2.14) and (2.15) we have
\[
(2.16) \quad fz = gz = hz = z = S(z, z, \cdots, z) = T(z, z, \cdots, z) = R(z, z, \cdots, z).
\]
Suppose there exists \( z' \in X \) such that
\[
fz' = gz' = hz' = z' = S(z', z', \cdots, z') = T(z', z', \cdots, z') = R(z', z', \cdots, z').
\]
Then
\[
D^*(z, z, z') = D^*(S(z, z, \cdots, z), T(z, z, \cdots, z), R(z', z', \cdots, z'))
\]
\[
\leq \lambda \max \left\{ D^*(gz, hz, fz'), D^*(hz, fz, gz'), \\ D^*(fz, gz, hz'), \cdots, D^*(gz, hz, fz'), \\ D^*(hz, fz, gz'), D^*(fz, gz, hz') \right\}
\]
\[
= \lambda D^*(z, z', z')
\]
which implies that \( z' = z \).

Thus \( z \) is the unique point in \( X \) satisfying (2.16).

Now we give an example to illustrate our main Theorem 2.1.

**Example 2.1.** Let \( X = [0, 1] \) and \( D^*(x, y, z) = |x - y| + |y - z| + |z - x| \) and \( k = 1 \). Define
\[
S(x, y, z) = \frac{2x + 3y^2 + 4z^3}{12}, \quad T(x, y, z) = \frac{3x^2 + 4y^3 + 2z}{12}, \quad R(x, y, z) = \frac{4x^3 + 2y + 3z^2}{12},
\]
\[
fz = \frac{x^2}{y}, \quad gz = \frac{z}{x}, \quad \text{and} \quad hz = \frac{z^2}{x}
\]
for all \( x, y, z, u, v, w, p, q, r \in X \). Consider
\[
D^*(S(x, y, z), T(u, v, w), R(p, q, r)) = \frac{2x^2 + 3y^2 + 4z^2}{72} - \frac{3u^2 + 4v^2 + 2w}{72} + \frac{3u^2 + 4v^2 + 2w}{72} - \frac{4p^2 + 2q + 3r^2}{72}
\]
\[
\leq \frac{1}{72} \left\{ \frac{2x^2 - 3u^2}{72} + \frac{3u^2 - 4p^2}{72} + \frac{4p^2 - 2x}{72} \right\}
\]
\[
\leq \frac{1}{6} \left\{ \frac{x^2}{4} - \frac{u^2}{4} + \frac{v^2}{4} + \frac{w^2}{4} \right\}
\]
\[
= \frac{1}{6} D^*(gx, hu, fp) + D^*(hy, fv, gq) + D^*(fz, gw, hr) \leq \frac{1}{2} \max\{D^*(gx, hu, fp), D^*(hy, fv, gq), D^*(fz, gw, hr)\}.
\]

Thus the condition (2.1.2) of Theorem 2.1 is satisfied. Clearly
\[
fz = S(x, x, \cdots, x),
\]
\[
gx = T(x, x, \cdots, x) and hx = R(x, x, \cdots, x)
\]

implies that \( x = 0 \) and
\[
f(S(0, 0, \cdots, 0)) = S(f(0, 0, \cdots, 0), g(T(0, 0, \cdots, 0)) = T(g(0, 0, \cdots, 0)
\]

and \( h(R(0, 0, \cdots, 0)) = R(h(0, 0, \cdots, 0). \) Hence the condition (2.1.3) is satisfied.

One can easily verify (2.1.1) and (2.1.4).

Clearly, 0 is the unique point in \( X \) such that
\[
f0 = g0 = h0 = 0 = S(0, 0, \ldots, 0, 0) = T(0, 0, \ldots, 0, 0) = R(0, 0, \ldots, 0, 0).
\]

**Corollary 2.1.** Let \((X, D^*)\) be a \(D^*\)-metric space, where \(D^*\) is of first type, \(k\) a positive integer and \(S, T, R : X^{3k} \rightarrow X\) and \(f : X \rightarrow X\) be mappings satisfying
\[
(2.1)^* \quad S(X^{3k}) \subseteq f(X), \quad T(X^{3k}) \subseteq f(X), \quad R(X^{3k}) \subseteq f(X),
\]
\[
(2.2)^* \quad D^* \left( \begin{array}{c}
S(x_1, x_2, \cdots, x_{3k}),
T(y_1, y_2, \cdots, y_{3k}),
R(z_1, z_2, \cdots, z_{3k})
\end{array} \right) \leq \lambda \max \{D^*(fx_i, fy_i, fz_i) / 1 \leq i \leq 3k\}
\]

for all \( x_1, x_2, \cdots, x_{3k}, y_1, y_2, \cdots, y_{3k}, z_1, z_2, \cdots, z_{3k} \in X \) and \( 0 < \lambda < 1 \),

(2.3)^* \quad One of the pairs \((S, f), (T, f)\) and \((R, f)\) is \(3k\)-weakly compatible,

(2.4)^* \quad \( f(X) \) is a complete subspace of \( X \).

Then there exists a unique \( z \in X \) such that
\[
fz = z = S(z, z, \cdots, z) = T(z, z, \cdots, z) = R(z, z, \cdots, z).
\]

**Corollary 2.2.** Let \((X, D^*)\) be a \(D^*\)-metric space, where \(D^*\) is of first type, \(k\) a positive integer and \(S : X^k \rightarrow X\) and \(f : X \rightarrow X\) be mappings satisfying
\[
(2.1)^* \quad S(X^k) \subseteq f(X), \quad R(X^k) \subseteq f(X),
\]
\[
(2.2)^* \quad D^* \left( \begin{array}{c}
S(x_1, x_2, \cdots, x_k),
S(y_1, y_2, \cdots, y_k),
S(z_1, z_2, \cdots, z_k)
\end{array} \right) \leq \lambda \max \{D^*(fx_i, fy_i, fz_i) / 1 \leq i \leq 3k\}
\]

for all \( x_1, x_2, \cdots, x_k, y_1, y_2, \cdots, y_k, z_1, z_2, \cdots, z_k \in X \) and \( 0 < \lambda < 1 \),
Theorem 2.2\hspace{1em} S(X^k) \subseteq f(X),

2.2.3\hspace{1em} The pair \((S, f)\) is \(k\)-weakly compatible,

2.2.4\hspace{1em} \(f(X)\) is a complete subspace of \(X\).

Then there exists a unique \(z \in X\) such that \(fz = z = S(z, z, \cdots, z)\).

\textbf{Corollary 2.3.} Let \((X, D^*)\) be a complete \(D^*\)-metric space, where \(D^*\) is of first type, \(k\) a positive integer and \(S, T, R : X^{3k} \to X\) be mappings satisfying

\[
D^* \begin{pmatrix} S(x_1, x_2, \cdots, x_{3k}) \\ T(y_1, y_2, \cdots, y_{3k}) \\ R(z_1, z_2, \cdots, z_{3k}) \end{pmatrix} \leq \lambda \max \{D^*(x_i, y_i, z_i) / 1 \leq i \leq 3k\}
\]

for all \(x_1, x_2, \cdots, x_{3k}, y_1, y_2, \cdots, y_{3k}, z_1, z_2, \cdots, z_{3k} \in X\) and \(0 < \lambda < 1\).

Then there exists a unique point \(z \in X\) such that \(z = S(z, z, \cdots, z) = T(z, z, \cdots, z) = R(z, z, \cdots, z)\).

\textbf{Corollary 2.4.} Let \((X, D^*)\) be a complete \(D^*\)-metric space, where \(D^*\) is of first type, \(k\) a positive integer and \(S : X^{3k} \to X\) be mappings satisfying

\[
D^* \begin{pmatrix} S(x_1, x_2, \cdots, x_{3k}) \\ S(y_1, y_2, \cdots, y_{3k}) \\ S(z_1, z_2, \cdots, z_{3k}) \end{pmatrix} \leq \lambda \max \{D^*(x_i, y_i, z_i) / 1 \leq i \leq 3k\}
\]

for all \(x_1, x_2, \cdots, x_{3k}, y_1, y_2, \cdots, y_{3k}, z_1, z_2, \cdots, z_{3k} \in X\) and \(0 < \lambda < 1\).

Then there exists a unique point \(z \in X\) such that \(z = S(z, z, \cdots, z)\).

\textbf{Remark 2.1.} Now we give probable modifications of Theorems of [5, 12, 13]:

(i) In Theorem 2.1 of [5], Theorem 3.1 of [12], Theorem 3.1 of [13] one has to assume \(G(x, x, y) \leq G(x, y, z)\) for all \(x, y, z \in X\), instead of \((G_4)\).

(ii) In (2.1.2) of Theorem 2.1 of [5], in (3.3) of Theorem 3.1 of [12] and in (5) of Theorem 3.1 of [13] one has to assume that any two of \(u, v, w\) are different instead of \(u \neq v \neq w\).

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\textbf{References}


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