PRIME BI-IDEALS IN $\Gamma$-SEMIRINGS

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Abstract. In this paper we introduce the concepts of prime, semiprime, strongly prime, irreducible and strongly irreducible bi-ideals in a $\Gamma$-semiring. Characterizations of a $\Gamma$-semiring using these concepts are furnished. A topology on the set of strongly prime bi-ideals is defined and a property of the space of strongly prime bi-ideals of a $\Gamma$-semiring is furnished.

1. Introduction

The notion of $\Gamma$-rings was introduced by Nobusawa in [10]. The class of $\Gamma$-rings contains not only all rings but also ternary rings. As a generalization of rings, semirings were introduced by Vandiver [14] and he obtained many results about it. Further as a generalization of $\Gamma$-rings and semirings, the notion of a $\Gamma$-semiring was introduced by Rao [11]. It is well known that ideals play an important role in any abstract algebraic structures. Characterizations of ideals in a semigroup were given by Lajos [8], while ideals in semirings were characterized by Iseki [4, 5]. Prime and semiprime ideals in $\Gamma$-semirings were discussed by Dutta and Sardar [2]. Authors were studied quasi-ideals and bi-ideals in $\Gamma$-semirings [6, 7]. The notion of a bi-ideal was first introduced for semigroups by Good and Hughes [3]. The concept of a bi-ideal for ring was given by Lajos [9] and for semirings by Shabir, Ali and Batool [12]. The concept of a bi-ideal in a semigroup (ring and semiring) is a generalization of one sided ideal and two sided ideal in a semigroup (ring and semiring). Prime bi-ideals in a $\Gamma$-ring was introduced by Booth and Groeneveld [1] and in a semigroup by Shabir and Kanwal [13].

In this paper efforts are made to extend the notion of prime ideals and semiprime ideal in $\Gamma$-semirings to prime bi-ideal and semiprime bi-ideal respectively in $\Gamma$-semirings. Also we define strongly prime bi-ideal in $\Gamma$-semirings and discuss some

2010 Mathematics Subject Classification. 16Y60, 16Y99.

Key words and phrases. Bi-ideal, prime bi-ideal, semiprime bi-ideal, strongly prime bi-ideal, irreducible bi-ideal, strongly irreducible bi-ideal.
properties of it. Finally we prove a topological property of the space of strongly prime bi-ideals of a $\Gamma$-semiring.

2. Preliminaries

First we recall some definitions of the basic concepts of $\Gamma$-semirings that we need in sequel. For this we refer Dutta and Sardar [2].

**Definition 2.1.** Let $S$ and $\Gamma$ be two additive commutative semigroups. $S$ is called a $\Gamma$-semiring if there exists a mapping $S \times \Gamma \times S \rightarrow S$ denoted by $a\alpha b$; for all $a, b \in S$ and $\alpha \in \Gamma$ satisfying the following conditions:

(i) $a\alpha (b + c) = (a\alpha b) + (a\alpha c)$

(ii) $(b + c)\alpha a = (b\alpha a) + (c\alpha a)$

(iii) $a(\alpha + \beta)c = (a\alpha c) + (a\beta c)$

(iv) $a\alpha (b\beta c) = (a\alpha b)\beta c$ ; for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Obviously, every semiring $S$ is a $\Gamma$-semiring.

Let $S$ be a semiring and $\Gamma$ be a commutative semigroup. Define a mapping $S \times \Gamma \times S \rightarrow S$ by, $a\alpha b = ab$; for all $a, b \in S$ and $\alpha \in \Gamma$. Then $S$ is a $\Gamma$-semiring.

**Definition 2.2.** An element $0 \in S$ is said to be an absorbing zero if $0a = 0 = a0$, $a + 0 = 0 + a = a$ ; for all $a \in S$ and $\alpha \in \Gamma$.

Now onwards $S$ denotes a $\Gamma$-semiring with absorbing zero unless otherwise stated.

**Definition 2.3.** A non-empty subset $T$ of $S$ is said to be a sub-$\Gamma$-semiring of $S$ if $(T, +)$ is a subsemigroup of $(S, +)$ and $a\alpha b \in T$; for all $a, b \in T$ and $\alpha \in \Gamma$.

**Definition 2.4.** A non-empty subset $T$ of $S$ is called a left (respectively right) ideal of $S$ if $T$ is a subsemigroup of $(S, +)$ and $\alpha a \in T$ (respectively $\alpha x \in T$) for all $a \in T$, $x \in S$ and $\alpha \in \Gamma$.

**Definition 2.5.** If $T$ is both left and right ideal of $S$, then $T$ is known as an ideal of $S$.

If $M, N$ are non-empty subsets of $S$, then

$M \Gamma N = \{ \sum_{i=1}^{n} x_i \alpha_i y_i \parallel x_i \in M, \alpha_i \in \Gamma, y_i \in N \}$. 

**Definition 2.6.** An element $a$ of a $\Gamma$-semiring $S$ is said to be regular if $a \in a\Gamma\Gamma a$.

If all elements of a $\Gamma$-semiring $S$ are regular, then $S$ is known as a regular $\Gamma$-semiring.

**Definition 2.7.** $S$ is said to be an intra-regular $\Gamma$-semiring if for any $x \in S$, $x \in S\Gamma x\Gamma xS$. 


Lemma 2.1. $S$ is regular if and only if $R \Gamma L = R \cap L$, for a right ideal $R$ and left ideal $L$ of $S$.

Lemma 2.2. Let $(a)_b$ denote the bi-ideal generated by $a \in S$. If $S$ is a regular $\Gamma$-semiring, then $(a)_b = a \Gamma S \Gamma a$.

3. Prime Bi-ideals

Here we recall the definition of a bi-ideal in $\Gamma$-semiring from [7].

Definition 3.1. A non-empty subset $B$ of $S$ is said to be a bi-ideal of $S$ if $B$ is a sub-$\Gamma$-semiring of $S$ and $B \subseteq S \subseteq B$.

Example 3.1. Consider the semiring $S = M_{2\times 2}(N_0)$, where $N$ denotes the set of all natural numbers and $N_0 = N \cup \{0\}$. If $\Gamma = S$, then $S$ forms a $\Gamma$-semiring with $A \alpha B = \text{usual matrix product of } A, \alpha, B$; for all $A, \alpha, B \in S$.

(1) $C = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \ | \ x, y \in N_0 \right\}$ is a bi-ideal of $S$.

(2) $D = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \ | \ x \in N_0 \right\}$ is a bi-ideal of $S$.

Statements given in the following theorem are easy to verify.

Theorem 3.1. In $\Gamma$-semiring $S$ following statements hold.

(1) Any one sided (two sided) ideal of $S$ is a bi-ideal of $S$.
(2) Intersection of a right ideal and a left ideal of $S$ is a bi-ideal of $S$.
(3) Arbitrary intersection of bi-ideals of $S$ is also a bi-ideal of $S$ and hence the set of all bi-ideals of $S$ forms a complete lattice.
(4) If $B$ is a bi-ideal of $S$, then $B \Gamma s$ and $s \Gamma B$ are bi-ideals of $S$, for any $s \in S$.
(5) If $B$ is a bi-ideal of $S$, then $b \Gamma B \Gamma c$ is a bi-ideal of $S$, for $b, c \in S$.
(6) If $B$ is a bi-ideal of $S$ and if $T$ is a sub-$\Gamma$-semiring of $S$, then $B \cap T$ is a bi-ideal of $T$.
(7) If $A, B$ are bi-ideals of $S$, then $A \Gamma B$ and $B \Gamma A$ are bi-ideals of $S$.
(8) For any $a \in S$, $S \Gamma a$ is a left ideal and $a \Gamma S$ is a right ideal of $S$.

Definition 3.2. A bi-ideal $B$ of $S$ is called a prime bi-ideal if $B_1 \Gamma B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$, for any bi-ideals $B_1$ and $B_2$ of $S$.

Definition 3.3. A bi-ideal $B$ of $S$ is called a strongly prime bi-ideal if $(B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$, for any bi-ideals $B_1$ and $B_2$ of $S$.

Definition 3.4. A bi-ideal $B$ of $S$ is called a semiprime bi-ideal if for any bi-ideal $B_1$ of $S$, $B_1 \Gamma B_1 \subseteq B$ implies $B_1 \subseteq B$.

Obviously every strongly prime bi-ideal in $S$ is a prime bi-ideal and every prime bi-ideal in $S$ is a semiprime bi-ideal.
Definition 3.5. A bi-ideal $B$ of $S$ is called an irreducible bi-ideal if $B_1 \cap B_2 = B$ implies $B_1 = B$ or $B_2 = B$, for any bi-ideals $B_1$ and $B_2$ of $S$.

Definition 3.6. A bi-ideal $B$ of $S$ is called a strongly irreducible bi-ideal if for any bi-ideals $B_1$ and $B_2$ of $S$, $B_1 \cap B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$.

Obviously every strongly irreducible bi-ideal is an irreducible bi-ideal.

Theorem 3.2. The intersection of any family of prime bi-ideals of $S$ is a semiprime bi-ideal.

Proof. Let $\{P_i| i \in \Lambda\}$ be the family of prime bi-ideals of $S$. For any bi-ideal $B$ of $S$, $B^2 \subseteq \bigcap_i P_i$ implies $B^2 \subseteq P_i$, for all $i \in \Lambda$. As $P_i$ are prime bi-ideals, $P_i$ are semiprime bi-ideals. Therefore $B \subseteq P_i$, for all $i \in \Lambda$. Hence $B \subseteq \bigcap_i P_i$. □

Theorem 3.3. Every strongly irreducible, semiprime bi-ideal of $S$ is a strongly prime bi-ideal.

Proof. Let $B$ be a strongly irreducible and semiprime bi-ideal of $S$. For any bi-ideals $B_1$ and $B_2$ of $S$, $(B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) \subseteq B$. Hence by Theorem 3.1(3), $B_1 \cap B_2$ is a bi-ideal of $S$. Since

$$(B_1 \cap B_2)^2 = (B_1 \cap B_2) \Gamma (B_1 \cap B_2) \subseteq B_1 \Gamma B_2.$$  

Similarly we get $(B_1 \cap B_2)^2 \subseteq B_2 \Gamma B_1$. Therefore

$$(B_1 \cap B_2)^2 \subseteq (B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) \subseteq B.$$  

As $B$ is a semiprime bi-ideal of $S$, $B_1 \cap B_2 \subseteq B$. But $B$ is a strongly irreducible bi-ideal. Therefore $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence $B$ is a strongly prime bi-ideal of $S$. □

Theorem 3.4. If $B$ is a bi-ideal of $S$ and $a \in S$ such that $a \notin B$, then there exists an irreducible bi-ideal $I$ of $S$ such that $B \subseteq I$ and $a \notin I$.

Proof. Let $B$ be the family of all bi-ideals of $S$ which contain $B$ but do not contain an element $a$. Then $B$ is a non-empty as $B \in B$. This family of all bi-ideals of $S$ forms a partially ordered set under the inclusion of sets. Hence by Zorn’s lemma, there exists a maximal bi-ideal say $I$ in $B$. Therefore $B \subseteq I$ and $a \notin I$. Now to show that $I$ is an irreducible bi-ideal of $S$. Let $C$ and $D$ be any two bi-ideals of $S$ such that $C \cap D = I$. Suppose that $C$ and $D$ both contain $I$ properly. But $I$ is a maximal bi-ideal in $B$. Hence we get $a \in C$ and $a \in D$. Therefore $a \in C \cap D = I$ which is absurd. Thus either $C = I$ or $D = I$. Therefore $I$ is an irreducible bi-ideal of $S$. □

Theorem 3.5. Any proper bi-ideal $B$ of $S$ is the intersection of all irreducible bi-ideals of $S$ containing $B$.
Suppose that \( a \in S \). Then \( 2a \subseteq S \). But always \( R \) be a left ideal of \( L \). Therefore by (2), be any two bi-ideals of \( a \). Similarly we have \( B_1 \cap B_2 \subseteq B_1 \Gamma B_2 \cap B_2 \Gamma B_1 \). Hence \( B_1 \cap B_2 \subseteq (B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) \). By Theorem 3.1(7), \( B_1 \Gamma B_2 \) and \( B_2 \Gamma B_1 \) are bi-ideals of \( S \). Therefore by Theorem 3.1(3), \((B_1 \Gamma B_2) \cap (B_2 \Gamma B_1)\) is a bi-ideal of \( S \). Hence by (2),

\[ (B_1 \cap B_2)^2 = B_1 \cap B_2 \cap B_1 \Gamma B_2 \subseteq B_1 \Gamma B_2 \cap (B_1 \cap B_2)^2 \subseteq (B_1 \cap B_2)(B_1 \cap B_2) \subseteq B_1 \Gamma B_2. \]
Similarly we show that 

\[(B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) = ((B_1 \Gamma B_2) \cap (B_2 \Gamma B_1)) \Gamma (B_1 \Gamma B_2) \cap (B_2 \Gamma B_1)).\]

\[(B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) \subseteq (B_1 \Gamma B_2) \Gamma (B_2 \Gamma B_1) \subseteq B_1 \Gamma S \Gamma B_1 \subseteq B_1.\]

Similarly we show that 

\[(B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) \subseteq B_2. \text{ Thus } (B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) \subseteq B_1 \cap B_2. \text{ Hence } B_1 \cap B_2 = (B_1 \Gamma B_2) \cap (B_2 \Gamma B_1).\]

(3) \(\Rightarrow\) (4) Let \(B\) be any bi-ideal of \(S\). Suppose that \(B\) is a strongly irreducible bi-ideal of \(S\) and \(B\) is a strongly prime bi-ideal of \(S\). Therefore by (3), we have 

\[B_1 = B_1 \cap B_1 = (B_1 \Gamma B_1) \cap (B_1 \Gamma B_1) = B_1 \Gamma B_1 \subseteq B.\]

Hence every bi-ideal of \(S\) is semiprime.

(4) \(\Rightarrow\) (5) Let \(B\) be a proper bi-ideal of \(S\). Hence by the Theorem 3.5, \(B\) is the intersection of all proper irreducible bi-ideals of \(S\) which contains \(B\). By assumption every bi-ideal of \(S\) is semiprime. Hence each proper bi-ideal of \(S\) is the intersection of irreducible semiprime bi-ideals of \(S\) which contain it.

(5) \(\Rightarrow\) (2) Let \(B\) be a bi-ideal of \(S\). If \(B^2 = S\), then clearly result holds. Suppose that \(B^2 \neq S\). Then \(B^2\) is a proper bi-ideal of \(S\). Hence by assumption, \(B^2\) is the intersection of irreducible semiprime bi-ideals of \(S\) which contain it. \(B^2 = \cap \{B_i/ B_i \text{ is an irreducible semiprime bi-ideal}\}\). As each \(B_i\) is a semiprime bi-ideal, \(B \subseteq B_i\), for all \(i\). Therefore \(B \subseteq \bigcap_i B_i = B^2\). \(B^2 \subseteq B\) always. Hence we have \(B^2 = B\).

Thus we get \(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2) \Rightarrow (1)\). Hence all the statements are equivalent.

\textbf{Theorem 3.7.} Let \(S\) be a regular and intra-regular \(\Gamma\)-semiring. Then for any bi-ideal \(B\) of \(S\), \(B\) is strongly irreducible bi-ideal if and only if \(B\) is strongly prime bi-ideal.

\textbf{Proof.} Let \(S\) be a regular and intra-regular \(\Gamma\)-semiring. Suppose that \(B\) is a strongly irreducible bi-ideal of \(S\). To show that \(B\) is a strongly prime bi-ideal of \(S\). Let \(B_1\) and \(B_2\) be any two bi-ideals of \(S\) such that \((B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) \subseteq B\). By Theorem 3.6, \((B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) = B_1 \cap B_2\). Hence \(B_1 \cap B_2 \subseteq B\). But \(B\) is a strongly irreducible bi-ideal of \(S\). Therefore \(B_1 \subseteq B\) or \(B_2 \subseteq B\). Thus \(B\) is a strongly prime bi-ideal of \(S\).

Conversely, suppose that \(B\) is a strongly prime bi-ideal of \(S\). Let \(B_1\) and \(B_2\) be any two bi-ideals of \(S\) such that \(B_1 \cap B_2 \subseteq B\) and \((B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) = B_1 \cap B_2 \subseteq B\). As \(B\) is a strongly prime bi-ideal, \(B_1 \subseteq B\) or \(B_2 \subseteq B\). Therefore \(B\) is a strongly irreducible bi-ideal of \(S\).

\textbf{Theorem 3.8.} Every bi-ideal of \(S\) is a strongly prime bi-ideal if and only if \(S\) is both regular and intra-regular and the set of bi-ideals of \(S\) is a totally ordered set under the inclusion of sets.

\textbf{Proof.} Suppose that every bi-ideal of \(S\) is a strongly prime bi-ideal. Then every bi-ideal of \(S\) is a semiprime bi-ideal. Hence by the Theorem 3.6, \(S\) is regular and intra-regular. To show that the set of bi-ideals of \(S\) is a totally ordered set under inclusion of sets. Let \(B_1\) and \(B_2\) be any two bi-ideals of \(S\) from the set of bi-ideals of \(S\). \(B_1 \cap B_2\) is also a bi-ideal of \(S\) (see Theorem 3.1(3)). Hence by
assumption \( B_1 \cap B_2 \) is a strongly prime bi-ideal of \( S \). Therefore by Theorem 3.6, 
\[(B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) = B_1 \cap B_2 \subseteq B_1 \cap B_2.\] Then \( B_1 \subseteq B_1 \cap B_2 \) or \( B_2 \subseteq B_1 \cap B_2 \).
Therefore \( B_1 \cap B_2 = B_1 \) or \( B_1 \cap B_2 = B_2 \). Thus either \( B_1 \subseteq B_2 \) or \( B_2 \subseteq B_1 \). This shows that the set of bi-ideals of \( S \) is a totally ordered set under inclusion of sets.

Conversely, suppose that \( S \) is regular, intra-regular and the set of bi-ideals of \( S \) is a totally ordered set under inclusion of sets. Let \( B \) be any bi-ideal of \( S \). \( B_1 \) and \( B_2 \) be any two bi-ideals of \( S \) such that \( (B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) \subseteq B \). By the Theorem 3.6, we have \( (B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) = B_1 \cap B_2 \). Therefore \( B_1 \cap B_2 \subseteq B \). But by assumption either \( B_1 \subseteq B_2 \) or \( B_2 \subseteq B_1 \). Hence \( B_1 \cap B_2 = B_1 \) or \( B_1 \cap B_2 = B_2 \). Therefore \( B_1 \subseteq B \) or \( B_2 \subseteq B \). Therefore \( B \) is a strongly prime bi-ideal of \( S \).

**Theorem 3.9.** If the set of bi-ideals of \( S \) is a totally ordered set under inclusion of sets, then every bi-ideal of \( S \) is a strongly prime if and only if every bi-ideal of \( S \) is prime.

**Proof.** Let the set of bi-ideals of \( S \) be a totally ordered set under inclusion of sets. As every strongly prime bi-ideal of \( S \) is prime, the proof of only if part is obvious.

Conversely, suppose that every bi-ideal of \( S \) is prime. Then every bi-ideal of \( S \) is semiprime. Hence by the Theorem 3.6, \( S \) is both regular and intra-regular. Again by Theorem 3.8, every bi-ideal of \( S \) is a strongly prime bi-ideal.

**Theorem 3.10.** If the set of bi-ideals of \( S \) is a totally ordered set under inclusion of sets, then \( S \) is both regular and intra-regular if and only if each bi-ideal of \( S \) is prime.

**Proof.** Let the set of all bi-ideals of \( S \) be a totally ordered set under inclusion of sets. Suppose \( S \) is both regular and intra-regular. Let \( B \) be any bi-ideal of \( S \). For any bi-ideals \( B_1 \) and \( B_2 \) of \( S \), \( B_1 \Gamma B_2 \subseteq B \). By the assumption we have either \( B_1 \subseteq B_2 \) or \( B_2 \subseteq B_1 \). Assume \( B_1 \subseteq B_2 \). Then \( B_1 \Gamma B_1 \subseteq B_1 \Gamma B_2 \subseteq B \). Hence by Theorem 3.6, \( B \) is a semiprime bi-ideal of \( S \). Therefore \( B_1 \subseteq B \). Hence \( B \) is a prime bi-ideal of \( S \).

Conversely, suppose that every bi-ideal of \( S \) is prime. Hence every bi-ideal of \( S \) is semiprime. Therefore by Theorem 3.6, \( S \) is both regular and intra-regular.

**Theorem 3.11.** Following statements are equivalents in \( S \).

1. The set of bi-ideals of \( S \) is totally ordered set under inclusion of sets.
2. Each bi-ideal of \( S \) is strongly irreducible.
3. Each bi-ideal of \( S \) is irreducible.

**Proof.** (1) \( \Rightarrow \) (2). Suppose that the set of bi-ideals of \( S \) is a totally ordered set under inclusion of sets. Let \( B \) be any bi-ideal of \( S \). To show that \( B \) is a strongly irreducible bi-ideal of \( S \). Let \( B_1 \) and \( B_2 \) be any two bi-ideals of \( S \) such that \( B_1 \cap B_2 \subseteq B \). But by the hypothesis we have either \( B_1 \subseteq B_2 \) or \( B_2 \subseteq B_1 \). Therefore \( B_1 \cap B_2 = B_1 \) or \( B_1 \cap B_2 = B_2 \). Hence \( B_1 \subseteq B \) or \( B_2 \subseteq B \). Thus \( B \) is a strongly irreducible bi-ideal of \( S \).

(2) \( \Rightarrow \) (3) Suppose that each bi-ideal of \( S \) is strongly irreducible. Let \( B \) be any bi-ideal of \( S \) such that \( B = B_1 \cap B_2 \), for any bi-ideals \( B_1 \) and \( B_2 \) of \( S \). Hence by
(2) we have \( B_1 \subseteq B \) or \( B_2 \subseteq B \). As \( B \subseteq B_1 \) and \( B \subseteq B_2 \), we have \( B_1 = B \) or \( B_2 = B \). Hence \( B \) is an irreducible bi-ideal of \( S \).

(3) \( \Rightarrow \) (1) Suppose that each bi-ideal of \( S \) is an irreducible bi-ideal. Let \( B_1 \) and \( B_2 \) be any two bi-ideals of \( S \). Then \( B_1 \cap B_2 \) is also a bi-ideal of \( S \) (see Theorem 3.1(3)). Hence \( B_1 \cap B_2 = B_1 \cap B_2 \) implies \( B_1 \cap B_2 = B_1 \) or \( B_1 \cap B_2 = B_2 \) by assumption. Therefore either \( B_1 \subseteq B_2 \) or \( B_2 \subseteq B_1 \). This shows that the set of bi-ideals of \( S \) is a totally ordered set under inclusion of sets.

**Theorem 3.12.** A prime bi-ideal \( B \) of \( S \) is a prime one sided ideal of \( S \).

**Proof.** Let \( B \) be a prime bi-ideal of \( S \). Suppose \( B \) is not a one sided ideal of \( S \). Therefore \( B \Gamma S \not\subseteq B \) and \( STB \not\subseteq B \). As \( B \) is a prime bi-ideal

\[
(B \Gamma S) \Gamma (STB) \not\subseteq B,
\]

which is a contradiction. Therefore \( B \Gamma S \subseteq B \) or \( STB \subseteq B \). Hence \( B \) is a prime one sided ideal of \( S \).

**Theorem 3.13.** A bi-ideal \( B \) of \( S \) is prime if and only if for a right ideal \( R \) and a left ideal \( L \) of \( S \), \( R \Gamma L \subseteq B \) implies \( R \subseteq B \) or \( L \subseteq B \).

**Proof.** Suppose that a bi-ideal of \( S \) is a prime bi-ideal of \( S \). Let \( R \) be a right ideal and \( L \) be a left ideal of \( S \) such that \( R \Gamma L \subseteq B \). Itself \( R \) and \( L \) are bi-ideals of \( S \) (see Theorem 3.1(2)). Hence \( R \subseteq B \) or \( L \subseteq B \). Conversely, we have to show that a bi-ideal \( B \) of \( S \) is a prime bi-ideal of \( S \). Let \( A \) and \( C \) be any two bi-ideals of \( S \) such that \( A \Gamma C \subseteq B \). For any \( a \in A \) and \( c \in C \), \( (a)_r \subseteq A \) and \( (c)_l \subseteq C \), where \( (a)_r \) and \( (c)_l \) denotes the right ideal and left ideal generated by \( a \) and \( c \) respectively. Therefore \( (a)_r \Gamma (c)_l \subseteq A \Gamma C \subseteq B \). Hence by the assumption, \( (a)_r \subseteq B \) or \( (c)_l \subseteq B \). Therefore \( a \in B \) or \( c \in B \). Thus \( A \subseteq B \) or \( C \subseteq B \). Hence \( B \) is a prime bi-ideal of \( S \).

**Theorem 3.14.** If \( B \) is a strongly irreducible bi-ideal of a regular and intra-regular \( \Gamma \)-semiring \( S \), then \( B \) is a prime bi-ideal.

**Proof.** Let \( B \) be a strongly irreducible bi-ideal of a regular and intra-regular \( \Gamma \)-semiring \( S \). Let \( B_1 \) and \( B_2 \) be any two bi-ideals of \( S \) such that \( B_1 \Gamma B_2 \subseteq B \). \( B_1 \cap B_2 \) is also a bi-ideal of \( S \) (see Theorem 3.1(3)). Therefore by Theorem 3.6, 

\[
(B_1 \cap B_2)^2 = (B_1 \cap B_2).
\]

Hence \( B_1 \cap B_2 = (B_1 \cap B_2)^2 = (B_1 \cap B_2) \Gamma (B_1 \cap B_2) \subseteq B_1 \Gamma B_2 \subseteq B \). As \( B \) is a strongly irreducible bi-ideal of \( S \), we have \( B_1 \subseteq B \) or \( B_2 \subseteq B \). Hence \( B \) is a prime bi-ideal of \( S \).

**4. Space of Strongly Prime Bi-ideals**

Let \( B \) be the family of all bi-ideals of \( S \). \( B \) is a partially ordered set under the inclusion of sets. Clearly \( B \) is a complete lattice under \( \lor \) and \( \land \) defined by

\[
B_1 \lor B_2 = B_1 + B_2 = (B_1 \cup B_2)_b \quad \text{and} \quad B_1 \land B_2 = B_1 \cap B_2, \quad \text{for all} \quad B_1, B_2 \in B.
\]

Let \( S \) be a \( \Gamma \)-semiring and \( \varphi_S \) be the set of all strongly prime bi-ideals of \( S \). For each bi-ideal \( B \) of \( S \) define
\[ \Theta_B = \{ J \in \varphi_S/B \nsubseteq J \} \text{ and } \zeta(\varphi_S) = \{ \Theta_B/B \text{ is a bi-ideal of } S \} \]

**Theorem 4.1.** If \( S \) is both regular and intra-regular, then \( \zeta(\varphi_S) \) forms a topology on the set \( \varphi_S \). There is an isomorphism between lattice of bi-ideals \( B \) and \( \zeta(\varphi_S) \), the lattice of open subsets of \( \varphi_S \).

**Proof.** Since \( \{0\} \) is a bi-ideal of \( S \) and each bi-ideal of \( S \) contains \( \{0\} \). Hence \( \Theta_{\{0\}} = \{ J \in \varphi_S/\{0\} \nsubseteq J \} = \Phi \). Therefore \( \Theta_{\{0\}} = \Phi \in \zeta(\varphi_S) \). As \( S \) itself bi-ideal, \( \Theta_S = \{ J \in \varphi_S/S \nsubseteq J \} = \varphi_S \) implies \( \varphi_S = \Theta_S \in \zeta(\varphi_S) \). Now let \( \Theta_{B_k} \in \zeta(\varphi_S) \), for \( k \in \Lambda \) (\( \Lambda \) is an indexing set) and \( B_k \) is a bi-ideal of \( S \). Therefore \( \Theta_{B_k} = \{ J \in \varphi_S/B_k \nsubseteq J \} \).

At the other hand, we have

\[ \bigcup_{k \in \Lambda} \Theta_{B_k} = \bigcup_{k \in \Lambda} \{ J \in \varphi_S/B_k \nsubseteq J \} = \{ J \in \varphi_S/B_k \nsubseteq J \text{ for some } k \in \Lambda \} \]

Hence

\[ \bigcup_{k \in \Lambda} \Theta_{B_k} = \{ J \in \varphi_S/\bigcup_{k \in \Lambda} B_k \nsubseteq J \} \]

where \( \bigcup_{k \in \Lambda} B_k \) denotes the bi-ideal of \( S \) generated \( \bigcup_{k \in \Lambda} B_k \). Therefore

\[ \bigcup_{k \in \Lambda} \Theta_{B_k} = \Theta_{\bigcup_{k \in \Lambda} B_k} \in \zeta(\varphi_S) \]

Further let \( \Theta_A, \Theta_B \in \zeta(\varphi_S) \). Let \( J \in \Theta_A \bigcap \Theta_B \) imply \( J \in \Theta_A \) and \( J \in \Theta_B \). Then \( A \nsubseteq J \) and \( B \nsubseteq J \). Suppose that \( A \bigcap B \subseteq J \). As \( S \) is both regular and intra-regular hence by the Theorem 3.6, \( A \bigcap B = (A \bigcap B) \bigcap (B \bigcap A) \). Therefore \( (A \bigcap B) \bigcap (B \bigcap A) \subseteq J \) and \( J \) is a strongly prime bi-ideal of \( S \) imply \( A \subseteq J \) or \( B \subseteq J \), which is a contradiction to \( A \nsubseteq J \) and \( B \nsubseteq J \). Hence \( A \bigcap B \nsubseteq J \) implies \( J \in \Theta_{A \bigcap B} \). Therefore \( \Theta_A \bigcap \Theta_B \subseteq \Theta_{A \bigcap B} \). Now let \( J \in \Theta_{A \bigcap B} \). Then \( A \bigcap B \nsubseteq J \) implies \( A \nsubseteq J \) and \( B \nsubseteq J \). Therefore \( J \in \Theta_A \) and \( J \in \Theta_B \) imply \( J \in \Theta_A \bigcap \Theta_B \). Thus \( \Theta_A \bigcap \Theta_B \subseteq \Theta_A \bigcap \Theta_B \). Therefore we get \( \Theta_A \bigcap \Theta_B = \Theta_{A \bigcap B} \in \zeta(\varphi_S) \). Hence \( \zeta(\varphi_S) \) forms a topology on the set \( \varphi_S \).

Now we define a function \( \phi : B \longrightarrow \zeta(\varphi_S) \) such that \( \phi(B) = \Theta_B \). Let \( A, B \in B \). Then

\[ \phi(A \bigcap B) = \Theta_{A \bigcap B} = \Theta_A \bigcap \Theta_B = \phi(A) \bigcap \phi(B) \]

and

\[ \phi(A + B) = \phi(A \bigcup B) = \Theta_{(A \bigcup B)_k} = \Theta_A \bigcup \Theta_B = \phi(A) \bigcup \phi(B) \]

Therefore \( \phi \) is a lattice homomorphism. Now let \( \phi(A) = \phi(B) \). Hence we have \( \Theta_A = \Theta_B \). Suppose that \( A \neq B \). Then there exists \( a \in A \) such that \( a \notin B \). As \( B \) is a proper bi-ideal of \( S \), by Theorem 3.4, there exists an irreducible bi-ideal \( J \) of \( S \) such that \( B \subseteq J \) and \( a \notin J \). By the Theorem 3.11, the set of all bi-ideals of \( S \) is totally ordered under inclusion of sets and also by the Theorem 3.8, \( J \) is a strongly prime bi-ideal of \( S \). Hence \( A \nsubseteq J \) or \( J \in \Theta_A \) implies \( B \nsubseteq J \). This contradicts to \( B \subseteq J \). Therefore \( A = B \). Hence \( \phi \) is a lattice isomorphism. \( \square \)
Remark 4.1. In the same way we can construct the space \( \mathcal{S} \) of strongly irreducible bi-ideals of \( S \).

Acknowledgement. Author is thankful for the learned referee for his valuable suggestions.

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Received by editors 04.06.2016; Revised version 05.10.2016; Available online 10.10.2016.

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