Construction and Finding the Area of the New Geometric Figure, \( n \)-cardioid

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Abstract – This paper is an original and basic research that will give the definition, the process of construction and the formula for finding the area of a new geometric figure formed on a mathematical problem of a goat tied in the corner of a shed. This new geometric figure was named as \( n \)-cardioid. The \( n \)-cardioid is the union of the arc \( X_i \) formed when a tension that measures half of the polygons perimeter is tied at any vertex of closed \( P_n \) and is rotated in a clockwise and counter clockwise direction until it touches the polygon, where \( P_n \) is a regular polygon with vertices \( x_1, x_2, \ldots, x_n \), \( n \) sides and length of sides \( s \) for all \( n \in \mathbb{N} \). The area of the \( n \)-cardioid will be solved using the formula

\[
A = \left( \frac{5n^2 + 4}{24} \right) s^2 \pi
\]

Keywords – \( n \)-cardioid, polygons, arc, area.

INTRODUCTION

Geometry as branch of mathematics deals with the deduction of the properties, measurement, and relationships of points, lines, angles, and figures in space. Its importance lies less in its results than in the systematic method Euclid used to develop and present them.

There are many geometric figures that can be seen in plant forms, in fabric designs, in art and in architecture.

One problem given to engineering students that challenge them in application of area of sectors caught the researcher’s attention. The problem says that “A goat is tied to a corner of a 4m by 5m shed by a 6m rope. What area of the ground can the goat graze?” The problem is solvable by applications of areas of sectors. The area of the ground that the goat can graze is the total area of the sectors. Geometrically, what if the shed is a square and the smaller sector has a radius that is equal to the length of the side of square. In doing so, the larger sector has a radius that is twice the smaller sector. By inspection, the radius of the larger sector is equal to the half of the perimeter of square. The researcher wants to extend the problem about it. What if the shed is a closed regular polygon and the rope measures half of its perimeter?

This study contributes a body of knowledge particularly in geometry. Also, it could be a way for exploring a new geometric figure and its properties. This research work could also significant to mathematics instructor, for this may help them to solve \( n \)-cardioid problems and other problems related to this. Finally, to the researcher, it develops logical skills and reasoning, wakes up a wide interest in field of mathematics especially in geometry.

OBJECTIVES OF THE STUDY

This study introduces a geometric figure called \( n \)-cardioid. Specifically it aims to give a definition of \( n \)-cardioid; discuss how to construct an \( n \)-cardioid; and generalize a formula for the area bounded by the \( n \)-cardioid.

MATERIALS AND METHODS

This is a basic research so the researcher collects and compiles data from mathematics books. It presents some properties of related geometric figures that are important in the introduction of a new geometric figure.

RESULTS AND DISCUSSION

Definition of \( n \)-cardioid

This section defines the \( n \)-cardioid and its parts.

**Definition 1.** Let \( P_n \) be a regular polygon with vertices \( x_1, x_2, \ldots, x_n \), \( n \) sides and length of sides \( s \) for all \( n \in \mathbb{N} \). Then there exist \( a_1, a_2, \ldots, a_m \) such that

\[
A = \left( \frac{5n^2 + 4}{24} \right) s^2 \pi
\]
Definition 2. Let \( P_n \) be a regular polygon with vertices \( x_1, x_2, \ldots, x_n \), \( n \) sides and length of sides \( s \) for all \( n \in \mathbb{N} \). Then there exist \( a_1, a_2, \ldots, a_m \) such that \( m \leq n \), the sector is said to be a polysector \( X_i \) if:

i. \( n \) is odd \( X_i = \{a_{i-1}x_{i}a_i; 2 \leq i \leq n \} \) where \( x_{i+1} \) lies on the line segment \( x_{i}a_i \) if \( i < \left\lfloor \frac{n}{2} \right\rfloor \) or \( x_{i-1} \) lies on the line segment \( x_{i}a_{i-1} \).

ii. \( n \) is even, then \( X_i = \left\{ \begin{array}{ll} a_{i-1}x_{i}a_i; 2 \leq i < \left\lfloor \frac{n}{2} \right\rfloor & \text{where} \\
_{2}x_{i}a_{i-1}; i = \frac{n}{2} & \\
_{2}x_{i}a_{i-1}; i = \frac{n}{2} + 2 & \\
_{2}x_{i}a_{i-1}; \left( \frac{n}{2} + 2 \right) < i \leq n & \\
_{i+1} \text{lies on the line segment} \ x_{i}a_{i} \text{for} & 2 \leq i < \frac{n}{2} \text{or} x_{i-1} \text{lies on line segment} \ x_{i}a_{i-1} \text{for} & \left( \frac{n}{2} + 2 \right) < i \leq n \end{array} \right. \)

iii. the angle of the sector is an exterior angle of the \( P_n \).

Remark 1 If \( n \) is even, there is no sector \( X_i \) formed when \( i = \left\lfloor \frac{n}{2} \right\rfloor \).

Definition 3. Let \( P_n \) be a regular with \( n \) sides and let \( X_i \) be the polysector, then the sequential sector is the set \( \bigcup_{i=2}^{n} X_i \).

Example 2. The figure below shows the sequential sector of the new geometric figures with octagon as its reference polygon. The octagon is just a reference in this figure and it does not belong to the sequential sector.

Definition 3.1.4. Let \( P_n \) be a regular with \( n \) sides. Let \( S_{X_i} \) denote the arc formed by the sector \( X_i \), then \( \bigcup_{i=1}^{n} S_{X_i} \) is called \( n \)-cardioid.

Example 3. The 8-cardioid forms when a 4-m tension is tied at any vertex of the closed regular octagon whose side’s measures one meter each and then moves counter clockwise and clockwise direction until it touches the part of the polygon as shown in the Figure 3.
This $n$-cardioid does not have sides, $n$ is just a reference to show where this geometric figure came from and $P_n$ does not belong to $n$-cardioid.

**Construction of $n$-cardioid**

Given a closed regular polygon $P_n$ with $n$ sides and length of sides $s$ and a tension that measures half of the regular polygon’s perimeter. The $n$-cardioid forms when a tension is tied at any vertex of the $P_n$. Then it moves counter clockwise and clockwise direction until it touches the part of the polygon.

**Finding the Area of $n$-cardioid**

The generalized formulas are being discussed in this section.

**Lemma 1.** Let $P_n$ be a regular with $n$ sides and length of each side $s$. Let $X_1$ be the reflexive sector, then area of the sector $X_1$ is given $A_{X_1} = \frac{n^2 + 2n}{8} s^2 \pi$.

*Illustration* Find the area of the reflexive sector or sector $X_1$ from a hexagon with 4 meter sides.

Solution

$$A_{X_1} = \frac{n^2 + 2n}{8} s^2 \pi$$

$$= \frac{6^2 + 2(6)}{8} 4^2 \pi$$

$$A = 96 \pi$$

The area of the $X_1$ is $96 \pi$ square meters.

**Lemma 2** Let $P_n$ be a regular with $n$ sides and length of each side $s$. Let $X_i$ be a polysector then the radius $X_i$ is given by $r = \left(\frac{ns}{2} - (i-1)s\right)$ for all $2 \leq i \leq \left[ \frac{n}{2} \right]$.

**Theorem 1.** Let $P_n$ be a regular with $n$ sides and length of each side $s$. Let $X_i$ be a polysector then, the area of each $X_i$ is given by $A = \frac{\pi}{n} \left(\frac{ns}{2} - (i-1)s\right)^2$ for all $2 \leq i \leq \left[ \frac{n}{2} \right]$.

**Remark 2.** Let $P_n$ be a regular with $n$ sides and let $X_i$ be the polysector with an angle that is equal to the exterior angle of the $P_n$ then, $X_2 = X_n, X_3 = X_{n-1}, ..., X_{\left[ \frac{n}{2} \right]} = X_{\left[ \frac{n}{2} \right]-1}$ such that $n \in \mathbb{N}$.

**Remark 3.** Let $P_n$ be a regular with $n$ sides and length of each side $s$. Let $A_{X_i}$ be the area of $X_i$. Since $\bigcup_{i=2}^{n} X_i$ be the sequential sector then, area of the sequential sector is $\sum_{i=2}^{n} A_{X_i}$.

**Lemma 3.** Let $P_n$ be a regular with $n$ sides and length of each side $s$. Let $\bigcup_{i=2}^{n} X_i$ be the sequential sector then, area of the sequential sector is given $A = \frac{(n-1)(n-2)}{12} s^2 \pi$.

*Illustration* Find the area of the sequential sector of the 8-cardioid whose sides of the reference polygon is 2 cm.

Solution:

$$A = \left(\frac{n^2 - 3n + 2}{12}\right) s^2 \pi$$

$$= \left(\frac{8^2 - 3(8) + 2}{12}\right) s^2 \pi$$

$$A = 14 \pi$$

The area of the sequential sector is $14 \pi$ square cm.
Remark 4. Let $P_n$ be a regular with $n$ sides and length of each side $s$. Let $A_i$ be the area of $X_i$. If $\bigcup_{i=1}^{n} X_i$ are the sectors formed by $n$-cardioid, then area of the $n$-cardioid is the $\sum_{i=1}^{n} A_i$.

**Theorem 2.** Let $P_n$ be a regular polygon with $n$ sides and let $s$ be the length of each sides. Let $\bigcup_{i=1}^{n} X_i$ are the sectors formed by $n$-cardioid, then the area of the $n$-cardioid is given by $A = \frac{5n^2 + 4}{24} s^2 \pi$.

**Illustration** Find the area of the 6-cardioid whose reference sides is 3 feet long.

**Solution**

$$A = \frac{5n^2 + 4}{24} s^2 \pi$$

$$= \frac{5(6)^2}{24} 3^2 \pi$$

$$= \frac{21}{2} \text{ft}^2$$

The area of the 6-cardioid is $\frac{21}{2} \text{ft}^2$.

**Theorem 3.** The area of the $n$-cardioid diverges.

**CONCLUSION AND RECOMMENDATION**

Through Mathematics we can simplify complicated problems. Instead of finding the area and arc length of each sectors formed, you can easily on finding the total area of all the sector forms when an object is tied to a corner of a closed regular polygon by a tension that is equal to the half of polygon’s perimeter. This object is rotate on the closed regular polygon. By observation while analyzing the problems and computing for the answers, you can find out a pattern that leads to come up with the generalize formula. Generalized formula on finding the area and the distance of the objects can graze is formed to simplify the computations.

The $n$-cardioid is still open for the other properties. Future researchers that may be done by other Mathematics practitioners may form a sequence from the area and circumlength of the $n$-cardioid. They may also find the longest line on the on the $n$-cardioid, then generate a formula. Finding the formula of the distance of the line that divides the $n$-cardioid on two equal parts given only the length of the sides and number of sides of the reference polygon.

**REFERENCES**


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