MAXIMUM INDEPENDENT SET COVER PEBBLING NUMBER OF COMPLETE GRAPHS AND PATHS

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Abstract. A pebbling move is defined by removing two pebbles from some vertex and placing one pebble on an adjacent vertex. A graph is said to be cover pebbled if every vertex has a pebble on it after a sequence of pebbling moves. The maximum independent set cover pebbling number, ρ(G), of a graph G is the minimum number of pebbles that are placed on V(G) such that after a sequence of pebbling moves, the set of vertices with pebbles forms a maximum independent set of G, regardless of their initial configuration. In this paper, we determine the maximum independent set cover pebbling number of complete graph and paths.

Keywords: Graph pebbling, cover pebbling, maximum independent set, complete graph and path.

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1. INTRODUCTION

Graphs considered here are simple, finite, undirected, and connected. Given a graph $G$, distribute $k$ pebbles on its vertices in some configuration. A pebbling move is defined by removing two pebbles from some vertex and placing one pebble on an adjacent vertex in which each move takes place along a path. The pebbling number [1], $\pi(G)$, of a graph $G$ is the minimum number of pebbles that are placed on $V(G)$, such that after a sequence of pebbling moves, a pebble can be moved to any root vertex $v$ in $G$ regardless of the initial configuration. One can find the survey of graph pebbling in [3]. The cover pebbling number [2], $\gamma(G)$ of a graph $G$ is defined as the minimum number of pebbles needed to place a pebble on every vertex of a graph $G$ using a sequence of pebbling moves, regardless of the initial configuration. A set $S$ of vertices in a graph $G$ is said to be independent set (or an internally stable set) if no two vertices in the set $S$ are adjacent. An independent set $S$ is maximum if $G$ has no independent set $S'$ with $|S'| > |S|$. In [4], we have introduced the concept of maximum independent set cover pebbling number. The maximum independent set cover pebbling number, $\rho(G)$ of a graph $G$, is the minimum number of pebbles that are placed on $V(G)$ such that after a sequence of pebbling moves, the set of vertices with pebbles forms a maximum independent set of $G$, regardless of their initial configuration. We have determined the maximum independent pebbling number of some families of graphs in [4, 5, 6]. In this paper, we determine the maximum independent set cover pebbling number $\rho(G)$ for complete graphs and path graphs.

Notation 1.1. For any vertex $a$ of $G$, $f(a)$ denotes the number of pebbles placed at the vertex $a$.

Notation 1.2. For $a, b \in V(G)$ and $ab \in E(G)$, $a \rightarrow b$ refers to moving $m$ pebbles to $b$ from $a$.

Notation 1.3. Throughout this paper we denote $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $P_n$ denotes the path $v_1, v_2, \ldots, v_n$.

2. MAXIMUM INDEPENDENT SET COVER PEBBLING NUMBER OF $K_n$ AND $P_n$

Let us now find the maximum independent set cover pebbling number of complete graph $K_n$. Clearly $\rho(K_1) = 1$. We may get a feeling that, when we place a pebble on any of the vertices of $K_n$, $n \geq 2$, we get a maximum independent set cover pebbling. Since the maximum independent set cover pebbling number is the least possible integer, it should imply that we should get a
maximum independent set cover pebbling for any number of pebbles greater than the maximum independent set cover pebbling number. In the case of the complete graph $K_n \ (n \geq 2)$, the number of pebbles between 2 and $n$ will not yield the property of maximum independent set cover pebbling.

Now we prove that $\rho (K_n) = n + 1$ for all $n \geq 2$.

**Theorem 2.1.** For $K_n$, $\rho (K_n) = n + 1 \ (n \geq 2)$.

**Proof.** Suppose a pebble is placed on each of the vertices $v_1, v_2, \ldots, v_n$ of $K_n$. Then we cannot cover a maximum independent set of $K_n$. Hence $\rho(K_n) > n$.

We use induction on $n$ to show that $\rho(K_n) \leq n + 1$. First we prove that the theorem is true for $n = 2$. Consider the distribution of three pebbles on the vertices of $K_2$. If we place three pebbles on a single vertex, say $v_1$, then we are done. Otherwise by pigeonhole principle, there exists a vertex, say $v_1$, with exactly two pebbles. Then moving a pebble to $v_2$ from $v_1$ covers a maximum independent set of $K_2$. Thus $\rho(K_n) \leq 3$. Now consider the distribution of $n + 1$ pebbles on the vertices of $K_n \ (n > 2)$. By pigeonhole principle, there exists a vertex, say $v_1$, with at least two pebbles.

**Case 1.** $f(v_1) = 2$.

We move a pebble to the vertex $v_2$ from $v_1$. Clearly $f(V(K_n)-\{v_1\}) = n$, and the induced sub graph of $V(K_n)-\{v_1\}$ is $K_{n-1}$. Hence we are done by induction.

**Case 2.** $f(v_1) > 2$.

In this case, it is easy to see that there is a vertex, say $v_i \ (i \neq 1)$ with zero pebbles. Then $f(V(K_n)-\{v_i\}) = n + 1$ and the induced sub graph of $V(K_n)-\{v_i\}$ is $K_{n-1}$. Hence we are done by induction. Thus $\rho(K_n) \leq n + 1$.

Let us now compute the maximum independent set cover pebbling number of a path $P_n$ on $n$ vertices. Since $P_2$ is isomorphic to $K_2$, $\rho (P_2) = 3$.

**Theorem 2.2.** For $P_3$, $\rho (P_3) = 6$.

**Proof.** Consider the following configuration: $f(v_2) = 5$ and $f(v_1) = f(v_3) = 0$. Then we cannot cover the maximum independent set of $P_3$. Thus $\rho (P_3) \geq 6$.

Now consider the distribution of six pebbles on the vertices of $P_3$. 

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Case 1. $1 \leq f(v_3) \leq 3$.

This implies that the path $v_1v_2$ contains at least three pebbles and hence we are done since $\rho(P_2) = 3$, except for the distribution $f(v_3) = 3$. In this case consider the following sequence of pebbling moves: $v_3 \rightarrow v_2 \rightarrow v_1$ and hence we are done.

Case 2. $f(v_3) = 0$.

We need at most four pebbles to put a pebble on $v_3$ from the vertices of $V(P_3) - \{v_3\}$. If we use three or four pebbles to pebble $v_3$ then we are done. Otherwise, $f(v_2) \geq 2$. If $f(v_2) = 2$ or $f(v_3) = 4$ then we are done by moving one or two pebbles to $v_3$ from $v_2$. If $f(v_2) = 6$ then we move a pebble to $v_1$ and two pebbles to $v_3$. For $f(v_2) = 3$ or $f(v_2) = 5$, we use two pebbles from $v_2$ to put a pebble at $v_3$. Then we can move all the pebbles from $v_2$ to $v_1$ using the pebbles at $v_1$ so that $v_1$ receives at least one pebble and hence we are done.

Case 3. $f(v_3) \geq 4$.

If $f(v_1) = 0$, then we apply Case 2. Let $f(v_1) \geq 1$. Then $f(v_2) \leq 1$. Suppose $f(v_2) = 0$. Then we are done. If $f(v_2) = 1$, then we move a pebble to $v_2$ from $v_3$ and then a pebble can be moved to $v_1$ from $v_2$ and hence we are done.

Thus $\rho(P_3) \leq 6$.

**Theorem 2.3.** For $P_4$, $\rho(P_4) = 6$.

**Proof.** Consider the following distribution: $f(v_1) = f(v_2) = 1$; $f(v_3) = 0$; $f(v_4) = 3$. Then we cannot cover a maximum independent set of $P_4$. Hence $\rho(P_4) > 5$. Now consider the distribution of six pebbles on the vertices of $P_4$. Let $P_A$ be the subgraph induced by the vertices $v_1$, $v_2$ and let $P_B$ be the subgraph induced by the vertices $v_3$, $v_4$. According to the distributions of six pebbles on the vertices of $P_A$ and $P_B$, we consider the following two cases:
1. Both $P_A$ and $P_B$ contain exactly three pebbles.
2. Any one of $P_A$ and $P_B$, say $P_A$, contains at most two pebbles.

**Case 1.** Both the paths $P_A$ and $P_B$ receive exactly three pebbles each.

Clearly we are done since $\rho(P_A) = \rho(P_B) = \rho(P_2) = 3$, except for the distribution $f(v_2) = f(v_3) = 3$. Now we consider the following pebbling moves: $v_3 \rightarrow v_2 \rightarrow v_1$ and hence we are done.

**Case 2.** Assume that $P_A$ contains at most two pebbles.

**Subcase 2.1.** Assume $f(P_A) = 0$.

Then $f(P_B) = 6$ and we are done since $f(P_B \cup \{v_2\}) = 6$ and $P_B \cup \{v_2\}$ is isomorphic to $P_3$. 
Subcase 2.2. Assume that $P_A$ has a pebble on it.

Then $P_B$ contains five pebbles. If $f(v_1) = 1$, then clearly we are done. So, assume that $f(v_2) = 1$. Then $f(P_B \cup \{v_2\}) = 6$ and $\rho(P_3) = 6$ and hence we are done.

Subcase 2.3. Assume that $P_A$ has two pebbles.

Then $P_B$ contains four pebbles. If $f(v_1) = 2$ or $f(v_2) = 2$, then clearly we are done. Let $f(v_1) = 1$ and $f(v_2) = 1$. If $f(v_3) \geq 2$, then we are done. If $f(v_3) = 1$, then $f(v_4) = 3$. Consider the following pebbling moves: $v_4 \rightarrow v_3 \rightarrow v_2 \rightarrow v_1$ and we are done. If $f(v_3) = 0$, then $f(v_4) = 4$. Consider the following pebbling moves: $v_4 \rightarrow v_3 \rightarrow v_2 \rightarrow v_3$ and hence we are done.

Thus $\rho(P_4) \leq 6$.

Theorem 2.4. For $P_5$, $\rho(P_5) = 21$.

Proof. Consider the following configuration: $f(v_5) = 20$, $f(v) = 0$ for all $v \in V(P_5) \setminus \{v_5\}$. Then we cannot cover the maximum independent set of $P_5$. Hence $\rho(P_5) > 20$. Let us consider the distribution of twenty one pebbles on the vertices of $P_5$ and different cases are discussed below.

For that, let $P_A$ be the subgraph induced by the vertices $v_1$ and $v_2$ and $P_B$ be the subgraph induced by the vertices $v_3$, $v_4$ and $v_5$.

Case 1. $f(P_A) \leq 2$.

Then $f(P_B) \geq 19$. Let $f(P_A) = 2$. If $f(v_1) = 2$ or $f(v_2) = 2$, then clearly we are done. So assume that $f(v_1) = 1$ and $f(v_2) = 1$. Using at most eight pebbles we can move a pebble to $v_2$ and then move a pebble to $v_1$. Hence the number of pebbles in $v_2$ is zero and we are done since $f(P_B) \geq 11$ and $\rho(P_B) = 6$. Let $f(P_A) = 1$. Clearly we are done if $f(v_1) = 1$. Assume that $f(v_2) = 1$. Using at most eight pebbles from $P_B$, we can move a pebble to $v_1$, so that $f(v_2)$ becomes zero. Hence we are done, since $f(P_B) \geq 12$ and $\rho(P_B) = 6$. Let $f(P_A) = 0$. Using at most sixteen pebbles we can place a pebble on $v_1$. Then $f(P_B) \geq 5$. If $f(P_B) \geq 6$, then clearly we are done since $\rho(P_B) = 6$. If $f(P_B) = 5$ then $f(v_5) = 5$ and hence we are done.

Case 2. Assume $f(P_B) \leq 5$.

Then $f(P_A) \geq 16$. If $3 \leq f(P_B) \leq 5$, then clearly we are done. Let $f(P_B) \leq 2$. This implies that $f(P_A) \geq 19$. If $f(P_B) = 2$ then also we are done. Assume that $f(P_B) = 1$. Using at most sixteen pebbles from $P_A$, we can put one pebble each on $v_3$ and $v_5$ so that $v_4$ has zero pebbles on it. Then $f(P_A) \geq 4$ and we are done. If $f(P_B) = 0$, then we can cover the maximum independent set of $P_5$ easily.
Case 3. Assume $f(P_A) \geq 3$ and $f(P_B) \geq 6$.

Clearly we are done except for the distribution $f(v_1) = 0$, $f(v_2) = 3$ and $f(P_B) = 18$. Using at most eight pebbles we can move a pebble to $v_2$ and then move two pebbles to $v_1$ from $v_2$. Hence we are done, since $\rho(P_3) = 6$ and $f(P_3) \geq 10$.

Thus $\rho(P_5) \leq 21$.

Theorem 2.5. For $P_6$, $\rho(P_6) = 21$.

Proof. Consider the following configuration: $f(v_6) = 20$, $f(v) = 0$ for all $V(G) \setminus \{v_6\}$. Then we cannot cover a maximum independent set of $P_6$. Hence $\rho(P_6) > 20$.

Now consider the distribution of twenty one pebbles on the vertices of $P_6$. Let $P_A$ be the subgraph induced by the vertices $v_1$ and $v_2$. Let $P_B$ be the subgraph induced by the vertices $v_3$, $v_4$, $v_5$ and $v_6$. According to the distribution of these twenty one pebbles on the vertices of $P_A$ and $P_B$, we find the following case:

Case 1. If $f(P_A) \leq 2$, then $f(P_B) \geq 19$.

Let $f(P_A) = 2$. If $f(v_1) = 2$ or $f(v_2) = 2$ then clearly we are done. So assume that $f(v_1) = 1$ and $f(v_2) = 1$. Then $f(P_B) = 19$. If $f(v_3) \geq 2$ then a pebble can be moved to $v_2$ from $v_3$ and then a pebble can be moved to $v_3$ from $v_2$. Then $f(P_B) \geq 18$ and we are done since $\rho(P_B) = \rho(P_4) = 6$. Assume $f(v_3) \leq 1$. If $f(v_3) = 1$, using at most eight pebbles from $P_B$ we can move a pebble to $v_3$. And from $v_3$, a pebble can be moved to $v_2$ and then we can move a pebble to $v_3$ from $v_2$. Now $f(P_B) \geq 12$ and hence we are done since $\rho(P_B) = \rho(P_4) = 6$. If $f(v_3) = 0$, then using at most sixteen pebbles from $P_B$ we can move a pebble to $v_3$. After moving a pebble to $v_2$ from $P_B$, if $f(P_B) \geq 6$ then we move a pebble to $v_1$ from $v_2$. If $3 \leq f(P_B) \leq 5$, then we move a pebble to $v_3$ from $v_2$. Clearly we are done, since $v_6$ is the only vertex contained $3$ or $4$ or $5$ pebbles on it. Let $f(P_A) = 1$. If $f(v_1) = 1$ then we are done since $f(P_B) = 20$ and $\rho(P_B) = 6$. Similarly we are done if $f(v_2) = 1$. Let $f(P_A) = 0$. Then $f(P_B) = 21$. Clearly we are done since $f(P_B \cup \{v_2\}) = 21$ and $\rho(P_5) = 21$.

Case 2. If $f(P_B) \leq 5$, then $f(P_A) \geq 16$.

If $3 \leq f(P_B) \leq 5$, then using at most twelve pebbles, we move at most three pebbles to $v_3$ from $P_A$, so $f(P_B) \geq 6$ and we are done. Then $f(P_A) \geq 4$, hence we can pebble the maximum
independent set of $P_A$, since $\rho(P_A) = 3$. If $f(P_B) = 2$, then $f(P_A) = 19$. Using at most sixteen pebbles from $P_A$ we can cover the maximum independent set of $P_6$. Assume that $f(P_B) = 1$. Then $f(P_A) = 20$. Suppose $f(v_6) = 0$ and $f(v_i) = 1$ for some $i = 3, 4, 5$. Then we are done since $\rho(P_6 - \{v_6\}) = \rho(P_5) = 21$. Suppose $f(v_6) = 1$ and $f(v_i) = 0$ for all $i = 3, 4, 5$. Then using six pebbles we can cover maximum independent set of $P_6 - \{v_5, v_6\}$ and we are done. If $f(P_B) = 0$, then we are done since $\rho(P_6 - \{v_6\}) = \rho(P_5) = 21$.

**Case 3.** $f(P_A) \geq 3$ and $f(P_B) \geq 6$.

Clearly we are done except for the distribution $f(v_1) = 0$, $f(v_2) = 3$ and $f(P_B) = 18$. Since $f(P_B \cup \{v_2\}) = 21$ and $\rho(P_5) = 21$, we are done in this distribution also.

Hence $\rho(P_6) \leq 21$.

**Theorem 2.6.** For $P_n (n \geq 5)$, $\rho(P_n) =$

\[
\begin{cases}
\frac{2^{n-1}}{3} & \text{if } n \text{ is even} \\
\frac{2^{n+1} - 1}{3} & \text{if } n \text{ is odd}
\end{cases}
\]

**Proof.** Consider the configuration where all pebbles are placed on the vertex $v_1$.

Clearly, we need at least

\[
\begin{cases}
\frac{2^{n-1}}{3} & \text{if } n \text{ is even} \\
\frac{2^{n+1} - 1}{3} & \text{if } n \text{ is odd}
\end{cases}
\]

pebbles to cover the maximum independent set $\{v_1, v_3, v_5, \ldots, v_{n-3}, v_{n-1}\}$ if $n$ is even

$\{v_1, v_2, v_3, \ldots, v_{n-2}, v_n\}$ if $n$ is odd

of $P_n$ from the vertex $v_1$.

Thus $\rho(P_n) \geq \begin{cases}
\frac{2^{n-1}}{3} & \text{if } n \text{ is even} \\
\frac{2^{n+1} - 1}{3} & \text{if } n \text{ is odd}
\end{cases}$.

Next we prove the upper bound by induction on $n$. The result is true for $n = 5$ and $n = 6$ from Theorem 2.4 and Theorem 2.5 respectively. Also note that

$\rho(P_m) = \rho(P_{m-1})$ when $m$ is even and for $n \geq 7$, $\rho(P_n) = \rho(P_{n-2}) + \begin{cases}
2^{n-2} & \text{if } n \text{ is even} \\
2^{n-1} & \text{if } n \text{ is odd}
\end{cases}$.

Now consider the distribution of $\rho(P_n)$ pebbles on the vertices of $P_n$.

**Case 1.** $n$ is odd.

Let $f(v_n) = 0$. If $f(v_{n-1}) = 0$, then we can pebble the vertex $v_n$ by using at most $2^{n-1}$ pebbles. Then the path $P_{n-2} : v_1v_2v_3 \ldots v_{n-3}v_{n-2}$ contains at least $\rho(P_{n-2})$ pebbles and hence we are done. So, assume that $f(v_{n-1}) \geq 1$. If $f(v_{n-1}) = 1$ or $3$ then also we are done. If $f(v_{n-1})$ is even then we move a
single pebble to \( v_n \) and then we move \( f(v_{n-1}) - 2 \) pebbles to \( v_{n-2} \). Hence, we are done since \( f(P_{n-2}) + \frac{f(v_{n-1}) - 2}{2} \geq \rho(P_{n-2}) \). If \( f(v_{n-1}) \geq 5 \) then consider the following sequence of pebbling moves: \( v_{n-1} \overset{2}{\rightarrow} v_n \rightarrow v_{n-1} \rightarrow v_n \) and then we move \( f(v_{n-1}) - 5 \) pebbles to \( v_{n-2} \). Here also, we are done since \( f(P_{n-2}) + \frac{f(v_{n-1}) - 5}{2} \geq \rho(P_{n-2}) \). So, we assume that \( f(v_n) \geq 1 \). In a similar way, we may assume that \( f(v_1) \geq 1 \). Consider the paths \( P_A : v_1v_2...v_{n-2} \) and \( P_B : v_3v_4...v_{n} \). Then, any one of the path contains at least \( \rho(P_{n-2}) \) pebbles. Without loss of generality, let \( P_A \) be the path. If \( f(v_{n-1}) = 0 \) then we are done easily. Similarly, we are done if \( f(v_{n-1}) + f(v_{n}) \geq 2 \) except the case \( f(v_n) = 1 \) and \( f(v_{n-1}) = 1 \). Now, we consider the case \( f(v_n) = 1 \) and \( f(v_{n-1}) = 1 \). For this case, \( P_A \) contains \( \rho(P_n) - 2 \) pebbles on it. Using at most \( 2^{n-2} \) pebbles, we can move a pebble to \( v_{n-1} \) from the vertices of \( P_A \) and then we move a pebble to \( v_{n-2} \) from \( v_{n-1} \). Thus, we are done since \( \rho(P_n) - 2^{n-2} \geq \rho(P_{n-2}) \).

**Case 2.** \( n \) is even.

Clearly, we are done if \( f(v_n) = 0 \) or \( f(v_1) = 0 \), since \( \rho(P_n) = \rho(P_{n-1}) \) when \( n \) is even. So, we assume that \( f(v_n) \geq 1 \) and \( f(v_1) \geq 1 \). Consider the paths \( P_A : v_1v_2...v_{n-2} \) and \( P_B : v_3v_4...v_{n} \). Then any one of the path contains at least \( \rho(P_{n-2}) \) pebbles. Without loss of generality, let \( P_A \) be the path. If \( f(v_{n-1}) = 0 \) then we are done easily. Similarly, we are done if \( f(v_{n-1}) + f(v_{n}) \geq 2 \) except the case \( f(v_n) = 1 \) and \( f(v_{n-1}) = 1 \). Finally, we consider the case \( f(v_n) = 1 \) and \( f(v_{n-1}) = 1 \). For this case, \( P_A \) contains \( \rho(P_n) - 2 \) pebbles on it. Using at most \( 2^{n-2} \) pebbles from the vertices of \( P_A \), we can move a pebble to \( v_{n-1} \). If we use exactly \( 2^{n-2} \) pebbles to pebble \( v_{n-1} \) from \( P_A \), then we move one pebble to \( v_{n-2} \) from \( v_{n-1} \) and hence we are done since the path \( v_1v_2...v_{n-4} \) contains more than \( \rho(P_{n-4}) \) pebbles. If we use less than \( 2^{n-2} \) pebbles to pebble \( v_{n-1} \) from \( P_A \), then we move one pebble to \( v_n \) from \( v_{n-1} \) and hence we are done since \( P_{n-2} \) contains at least \( \rho(P_{n-2}) \) pebbles.

Thus \( \rho(P_n) \leq \begin{cases} \frac{2^n - 1}{3} & \text{if } n \text{ is even} \\ \frac{2^{n+1} - 1}{3} & \text{if } n \text{ is odd} \end{cases} \).
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