



## DEVELOPMENT OF LOVEJOY AND OSBURN'S OVERPARTITION FUNCTIONS

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### Abstract:

*In 2008, Lovejoy and Osburn defined the generating function for  $\bar{P}(n)$ . In 2009, Byungchan Kim defined the generating function for  $P_2(n)$ . This paper shows how to discuss the generating functions for  $\bar{P}(n)$  and  $P_2(n)$ . Byungchan Kim also defined  $P_k(n)$  with increasing relation and overpartition congruences mod 4, 8 and 64. In 2006, Berndt found the relation  $d_{1,4}(n) - d_{3,4}(n)$  has two values with certain restrictions and various formulae by the common term  $\sigma(n)$ . This paper shows how to prove the four Theorems about overpartitions modulo 8. These Theorems satisfy the arithmetic properties of the overpartition function modulo 8.*

### Keywords:

*Convenience, congruent, modulo 8, prime factorizations, parity.*

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## 1. INTRODUCTION

In this paper we give some related definitions of overpartition,  $\omega(\lambda)$ ,  $P_k(n)$ ,  $d(n)$ ,  $d_{i,4}(n)$ ,  $\sigma(n)$  and  $\chi(n)$ . We discuss the generating functions for  $\bar{P}(n)$  and  $P_2(n)$ . We analyze

various relations  $\bar{P}(n) = \sum_k 2^k p_k(n)$ ,  $\bar{P}(3n+2) \equiv 0 \pmod{4}$ ,  $\bar{P}(4n+3) \equiv 0 \pmod{8}$ ,

$\bar{P}(8n+7) \equiv 0 \pmod{64}$ ,

$$d_{1,4}(n) - d_{3,4}(n) = \begin{cases} (r_1 + 1) \dots (r_k + 1), & \text{if } s_i \text{'s are even integers,} \\ 0, & \text{otherwise} \end{cases},$$

$\bar{P}(n) \equiv 2d_{1,4}(n) - 2d_{3,4}(n) - 2\chi(n) - 2\sigma(n) + 4d(n) \pmod{8}$ ,

$$\sigma(n) = (2^{a+1} - 1) \prod_i \left( \sum_{m=0}^{r_i} p_i^m \right) \prod_j \left( \sum_{m=0}^{s_j} q_j^m \right), \text{ and } \sigma(n) \equiv \begin{cases} (r_1 + 1) \dots (r_k + 1) \pmod{4} & \text{if } a = 0 \\ 3(r_1 + 1) \dots (r_k + 1) \pmod{4} & \text{otherwise,} \end{cases}$$



respectively. We prove the four Theorems about overpartitions modulo 8 with certain conditions of n.

**2. SOME RELATED DEFINITIONS**

Overpartition: An overpartition of n is a partition of n in which the first occurrence of a part may be overlined. Let  $\overline{P}(n)$  denote the number of overpartitions of an integer n. For convenience, define

$\overline{P}(0)=1$ . For example

n	$\overline{P}(n)$
1 : 1, $\overline{1}$	2
2 : 2, $\overline{2}$ , 1+1, $\overline{1}$ +1	4
3 : 3, $\overline{3}$ , 2+1, $\overline{2}$ +1, 2+ $\overline{1}$ , $\overline{2}$ + $\overline{1}$ , 1+1+1, $\overline{1}$ +1+1	8
4 : 4, $\overline{4}$ , 3+1, $\overline{3}$ +1, 3+ $\overline{1}$ , $\overline{3}$ + $\overline{1}$ , 2+2, $\overline{2}$ +2, 2+1+1, $\overline{2}$ +1+1, 2+ $\overline{1}$ +1, $\overline{2}$ + $\overline{1}$ +1, 1+1+1+1, $\overline{1}$ +1+1+1	14
...	...

Similarly we get;

$\overline{P}(5)= 24, \overline{P}(6)= 40, \overline{P}(7)= 64,...$

$\omega(\lambda)$  : An ordinary partition  $\lambda$ , there are  $2^{\omega(\lambda)}$  distinct overpartitions, where  $\omega(\lambda)$  is the number of distinct parts in  $\lambda$ . For example if  $\lambda = 2+1+1$ ;  $\omega(\lambda) = 2$ , there are four overpartitions [ $2+1+1, \overline{2}+1+1, 2+\overline{1}+1, \overline{2}+\overline{1}+1$ ] then.  $2^{\omega(\lambda)} = 2^2 = 4$ .

$P_k(n)$  [Byungchan Kim(2009)] : The number of partitions of n such that the number of distinct parts is exactly k. For example  $P_2(6)= 6$  since there are six partitions like 5+1, 4+2, 4+1+1, 3+1+1+1, 2+2+1+1, 2+1+1+1+1.

$d(n)$  : The number of the divisors of n.

$d_{i,4}(n)$  [Alladi (1997)] :The number of the divisors of n which are congruent to i modulo 4.

$\sigma(n)$  :The sum of the divisors of n.

$\chi(n)$  : The term is defined by  $\chi(n) = \begin{cases} 1, & \text{where n is a square of an integer,} \\ 0, & \text{otherwise.} \end{cases}$

For example,  $\chi(6) = 0, \chi(9) = 1,.....$

The generating function [Byungchan Kim(2009)] for  $\overline{P}(n)$  is given by



$$\begin{aligned}
 \prod_{n=1}^{\infty} \frac{(1+x^n)}{(1-x^n)} &= \frac{(1+x)(1+x^2)(1+x^3)\dots}{(1-x)(1-x^2)(1-x^3)\dots} \\
 &= (1+x+x^2+2x^3+2x^4+3x^5+\dots)(1+x+x^2+3x^3+5x^4+\dots) \\
 &= 1+2x+3x^2+8x^3+14x^4+24x^5+40x^6+64x^7+\dots \\
 &= \bar{P}(0) + \bar{P}(1)x + \bar{P}(2)x^2 + \bar{P}(3)x^3 + \bar{P}(4)x^4 + \dots \\
 &= \sum_{n=0}^{\infty} \bar{P}(n)x^n.
 \end{aligned}$$

The generating function [Byungchan Kim(2009)] for  $P_2(n)$  is given by

$$\begin{aligned}
 &\left( \sum_{k \geq 1} \frac{x^k}{1-x^k} \right)^2 - \sum_{k \geq 1} \left( \frac{x^k}{1-x^k} \right)^2 \\
 &= \left( \frac{x}{1-x} + \frac{x^2}{1-x^2} + \dots \right)^2 - \left[ \left( \frac{x}{1-x} \right)^2 + \left( \frac{x^2}{1-x^2} \right)^2 + \dots \right] \\
 &= 2 \frac{x}{1-x} \cdot \frac{x^2}{1-x^2} + 2 \frac{x}{1-x} \cdot \frac{x^3}{1-x^3} + 2 \frac{x}{1-x} \cdot \frac{x^4}{1-x^4} + \dots \\
 &= 2x^3(1+x+x^2+\dots)(1+x^2+\dots) + 2x^4(1+x+\dots)(1+x^3+\dots) + \dots \\
 &= 2x^3 + 4x^4 + 10x^5 + \dots \\
 &= 2P_2(3)x^3 + 2P_2(4)x^4 + 2P_2(5)x^5 + \dots \\
 &= \sum_{n \geq 1} 2P_2(n)x^n. \text{ For convenience } P_2(1) = 0 \text{ and } P_2(2) = 0.
 \end{aligned}$$

### 3. VARIOUS RELATIONS ABOUT OVERPARTITIONS

A) If  $n = 6$ ,  $\bar{P}(6) = 40$ ,  $P_1(6) = 4$  (like : 6, 3+3, 2+2+2, 1+1+1+1+1+1),  $P_2(6) = 6$ , and  $P_3(6) = 1$

$$\begin{aligned}
 \therefore 2P_1(6) + 2^2 P_2(6) + 2^3 P_3(6) \\
 &= 2.4 + 4.6 + 8.1 \\
 &= 8 + 24 + 8 = 40 = \bar{P}(6)
 \end{aligned}$$

$$\therefore \bar{P}(6) = 2P_1(6) + 2^2 P_2(6) + 2^3 P_3(6).$$

So we can write  $\bar{P}(n) = \sum_k 2^k P_k(n)$  [Andrews (1967)].

Reducing this modulo 8, we obtain  $\bar{P}(n) \equiv 2P_1(n) + 2^2 P_2(n) \pmod{8}$ , it is seen that  $P_1(n) = d(n)$ , when  $d(n)$  is the number of the divisors of  $n$  including 1 and  $n$ .

B) We get;



$$\bar{P}(2) = 4, \bar{P}(5) = 24, \dots \text{ i.e., } \bar{P}(2) = 4 \equiv 0 \pmod{4}, \bar{P}(3+2) = 24 \equiv 0 \pmod{4}, \dots$$

We can conclude that  $\bar{P}(3n+2) \equiv 0 \pmod{4}$ .

C) We get;

$$\bar{P}(3) = 8, \bar{P}(7) = 64, \dots \text{ i.e. } \bar{P}(3) = 8 \equiv 0 \pmod{8}, \bar{P}(4+3) = 64 \equiv 0 \pmod{8}.$$

We can conclude that  $\bar{P}(4n+3) \equiv 0 \pmod{8}$ .

D) We get;

$$\bar{P}(7) = 64, \bar{P}(15) = 1408, \dots \text{ i.e. } \bar{P}(7) = 64 \equiv 0 \pmod{64}, \bar{P}(8+7) = 1408 \equiv 0 \pmod{64}, \dots$$

We can conclude that  $\bar{P}(8n+7) \equiv 0 \pmod{64}$ . [Lovejoy et al (2008)]

E) Let  $n = 2^a p_1^{r_1} \dots p_k^{r_k} q_1^{s_1} \dots q_l^{s_l}$ ,

If  $d_{i,4}(n)$  is the number of the divisors which are congruent to  $i$  modulo 4.

Now if  $n = 9 = 3^2 = 3^{s_1}$  when  $s_1 = 2$  is the even integer

$$\therefore d_{1,4}(9) = 2, d_{3,4}(9) = 1, \text{ then } d_{1,4}(9) - d_{3,4}(9) = 2 - 1 = 1.$$

Again if  $n = 6 = 2 \cdot 3 = 2^a \cdot 3^{s_1}$  when  $a=1$  and  $s_1 = 1$

$$\therefore d_{1,4}(6) = 1, d_{3,4}(6) = 1,$$

$$\text{Then } d_{1,4}(6) - d_{3,4}(6) = 1 - 1 = 0.$$

We can conclude that if  $n$  has the prime factorization  $2^a p_1^{r_1} \dots p_k^{r_k} q_1^{s_1} \dots q_l^{s_l}$ , where the  $p_i$ 's are primes congruent to 1 modulo 4 and  $q_j$ 's are primes congruent to 3 modulo 4, then

$$d_{1,4}(n) - d_{3,4}(n) = \begin{cases} (r_1 + 1) \dots (r_k + 1), & \text{if } s_i \text{'s are even integers} \\ 0, & \text{otherwise [Fortin et al (2005)].} \end{cases}$$

F) We get;  $d_{1,4}(9) = 2$  (like, the divisors are 1 and 9)

$$d_{3,4}(9) = 1, \text{ ( like, the divisor is 3)}$$

Now we get;  $2 d_{1,4}(9) - 2 d_{3,4}(9) - 2\chi(9) - 2\sigma(9) + 4d(n)$

$$= 2 \times 2 - 2 \times 1 - 2 \times 1 - 2 \times 13 + 4 \times 3$$

$$= -14 \equiv 2 \pmod{8}, \text{ but } \bar{P}(9) = 154 \equiv 2 \pmod{8}.$$

$$\therefore \bar{P}(9) \equiv 2 d_{1,4}(9) - 2 d_{3,4}(9) - 2\chi(9) - 2\sigma(9) + 4d(9) \pmod{9}.$$

We can conclude that,  $\bar{P}(n) \equiv 2 d_{1,4}(n) - 2 d_{3,4}(n) - 2\chi(n) - 2\sigma(n) + 4d(n) \pmod{8}$ . [Byungchan Kim(2009)]

G) If  $n = 10 = 2 \cdot 5 = 2^a 5^{r_1}$  where  $a = 1$  and  $r_1 = 1$

$$\therefore \sigma(10) = (2^2 - 1)(5^0 + 5^1) \text{ but } \sigma(10) = \frac{2^2 - 1}{2 - 1} \cdot \frac{5^2 - 1}{5 - 1}$$



$$\begin{aligned}
 &= 3(1+5) &&= \frac{3}{1} \cdot \frac{24}{4} \\
 &= 18 &&= 18
 \end{aligned}$$

We can conclude that

$$\sigma(n) = (2^{a+1} - 1) \prod_i \left( \sum_{m=0}^{r_i} p_i^m \right) \prod_j \left( \sum_{m=0}^{s_j} q_j^m \right). \text{ [Andrews (1967)]}$$

H) We get,  $n = 9 = 3^2 = 2^a \cdot 3^{s_1}$  where  $a = 0$  and  $s_1 = 2$

$$\sigma(9) = \frac{3^2 - 1}{3 - 1} = \frac{26}{2} = 13 \equiv 1 \pmod{4} = (0+1) \equiv (r_1 + 1) \pmod{4},$$

again if  $n = 10 = 2 \cdot 5 = 2^a \cdot 5^{s_1}$  where  $a = 1$  and  $r_1 = 1$

$$\begin{aligned}
 \sigma(10) &= \frac{2^2 - 1}{2 - 1} \cdot \frac{5^2 - 1}{5 - 1} = 3 \cdot \frac{24}{4} = 18 \equiv 2 \pmod{4} \equiv 6 \pmod{4} = 3 \cdot 2 = 3(1+1) \\
 &\equiv 3(r_1 + 1) \pmod{4}.
 \end{aligned}$$

We can write that

$$\sigma(n) \begin{cases} \equiv (r_1 + 1) \dots (r_k + 1) \pmod{4} & \text{if } a = 0 \\ \equiv 3(r_1 + 1) \dots (r_k + 1) \pmod{4}, & \text{otherwise. [Fortin et al (2005)]} \end{cases}$$

#### 4. THEOREM

Let  $n$  be an integer, then

- 1)  $\bar{P}(n) \equiv 0 \pmod{8}$ , where  $n$  is not a square of an odd integer or an even integer and is not a double of a square.
- 2)  $\bar{P}(n) \equiv 2 \pmod{8}$ , if  $n$  is a square of an odd integer.
- 3)  $\bar{P}(n) \equiv 4 \pmod{8}$ , if  $n$  is a double of a square
- 4)  $\bar{P}(n) \equiv 6 \pmod{8}$ , if  $n$  is a square of an integer.

Proof: From above we get;

$$\chi(n) \begin{cases} \neq 1, & \text{when } n \text{ is a square of an integer,} \\ 0, & \text{otherwise.} \end{cases}$$

$$\bar{P}(n) \equiv 2(d_{1,4}(n) - d_{3,4}(n)) - 2\chi(n) - 2\sigma(n) + 4d(n) \pmod{8}, \dots (1)$$

where,  $d_{i,4}(n)$  is the number of the divisors which are congruent to  $i$  modulo 4.

Now we will consider the three cases according to the parity of  $r_i$  and  $s_j$

**Case 1:** There is an  $s_j$  that is odd and  $r_i$  is any integer, then

$$d_{1,4}(n) - d_{3,4}(n) = 0 \quad \chi(n) = 0 \quad \text{and} \quad d(n) \equiv 0 \pmod{8}.$$

From (1), we get



$$\bar{P}(n) \equiv 2 (d_{1,4}(n) - d_{3,4}(n)) - 2\chi(n) - 2\sigma(n) + 4d(n) \pmod{8},$$

$$\equiv 0 - 2 \times 0 - 2\sigma(n) + 0 \pmod{8}$$

$$\text{or } \bar{P}(n) \equiv -2\sigma(n) \pmod{8} \dots (2)$$

[since if  $n = 6 = 2 \cdot 3 = 2^a \cdot 3^{s_1}$  where  $a = 1$  and  $s_1$  in an odd integer, then

$$d_{1,4}(6) - d_{3,4}(6) = 0 \text{ and } \chi(6) = 0 \text{ where } 6 \text{ is not a square,}$$

$$\text{and } d(6) = d(2 \cdot 3) = (1+1)(1+1) = 4$$

$$\therefore 4d(6) = 4 \cdot 4 = 16 \equiv 0 \pmod{8}].$$

From relation G) we get;

$$\sigma(n) = (2^{a+1} - 1) \prod_i \left( \sum_{m=0}^{r_i} p_i^m \right) \prod_j \left( \sum_{m=0}^{s_j} q_j^m \right) \text{ [Berndt(2006)]}$$

$$\text{[since } s_j \text{'s are odd integers, so } \sum_{m=0}^{s_j} q_j^m \equiv 0 \pmod{4}].$$

$$\therefore \sigma(n) \equiv 0 \pmod{4} \text{ and } 2\sigma(n) \equiv 0 \pmod{8}].$$

From (2) we can conclude that  $\bar{P}(n) \equiv 0 \pmod{8}$  for such  $n$ .

**Case 2:** All  $s_j$ 's are even and there is an  $r_i$  that is odd.

Then,  $d_{1,4}(n) - d_{3,4}(n) = (r_1 + 1) \dots (r_k + 1)$ ,  $\chi(n) = 0$  where  $n$  is not a square and

$$4d(n) \equiv 0 \pmod{8}.$$

From (1) we get;

$$\bar{P}(n) \equiv 2(r_1 + 1) \dots (r_k + 1) - 2\sigma(n) \pmod{8} \dots (3)$$

$$\text{[ since if } n = 5 \cdot 3^2 = 45 \text{ } d(45) = d(5 \cdot 3^2) = (1+1) \cdot (2+1) = 2 \cdot 3 = 6$$

$$\therefore 4d(45) = 4 \cdot 6 = 24 \equiv 0 \pmod{8}].$$

and  $\sigma(n) \equiv (r_1 + 1) \dots (r_k + 1) \pmod{4}$ , where  $s_j$ 's are even  $r_i$ 's are odd and  $a = 0$ .

From (3), we can conclude that

$\bar{P}(n) \equiv 0 \pmod{8}$ , where  $n$  is not a square of an odd integer or an even integer and is not a double of a square. Hence the Theorem 1.

[Numerical example 1: If  $n$  is not a square of an odd integer or an even integer and is not a double of a square. We get;  $\bar{P}(3) = 8$ ,  $\bar{P}(5) = 24$ , ...

$$\therefore \bar{P}(3) = 8 \equiv 0 \pmod{8}, \bar{P}(5) = 24 \equiv 0 \pmod{8}, \dots$$

We can conclude that  $\bar{P}(n) \equiv 0 \pmod{8}$ , for such  $n$ .]

**Case 3:** All the  $r_i$ 's and  $s_j$ 's are even.

Suppose that  $a$  is 0. Then  $n$  is a square.

By (1) we deduce that



$$\bar{P}(n) \equiv 2 (r_1 + 1) \dots (r_k + 1) - 2 \sigma(n) + 4 \pmod{8} \dots (4)$$

[since  $d_{1,4}(n) - d_{3,4}(n) = (r_1 + 1) \dots (r_k + 1)$  where  $s_j$ 's are even and  $\chi(n) = 1$  where  $n$  is a square of an integer,  $d(n) \equiv 1 \pmod{8}$  and also,  $\sigma(n) \equiv (r_1 + 1) \dots (r_k + 1) \pmod{4}$  where  $r_i$ 's and  $s_j$ 's are even and also  $a = 0$ ]

From (4) we get;  $\bar{P}(n) \equiv 2 (r_1 + 1) \dots (r_k + 1) + 2 - 2 (r_1 + 1) \dots (r_k + 1) \pmod{8}$ .

$\therefore \bar{P}(n) \equiv 2 \pmod{8}$ , when  $n$  is a square of an odd integer. Hence the Theorem 2 .

[Numerical example 2: If  $n$  is not a square of an odd integer,

We get;  $\bar{P}(1) = 2$ ,  $\bar{P}(9) = 154$ , ...

$$\therefore \bar{P}(1) = 2 \equiv 2 \pmod{8}, \bar{P}(9) = 154 \equiv 2 \pmod{8}, \dots$$

We can conclude that  $\bar{P}(n) \equiv 2 \pmod{8}$ , for such  $n$ .]

Suppose that  $a$  is odd. Then  $n$  is a double of square.

From (1) we get;

$$\bar{P}(n) \equiv 2 (r_1 + 1) \dots (r_k + 1) - 2 \sigma(n) \pmod{8} \text{ [Berndt(2006)]}.$$

[since  $d_{1,4}(n) - d_{3,4}(n) = (r_1 + 1) \dots (r_k + 1)$  where  $r_i$ 's and  $s_j$ 's are even integers.

$\chi(n) = 0$ , where  $n$  is not a square of an integer.

If  $n = 2 \cdot 3^2 \cdot 5^2$

$$\therefore d(n) = (1+1) \cdot (2+1) \cdot (2+1) = 18$$

$$\therefore 4d(n) = 4 \cdot 18 = 72 \equiv 0 \pmod{8}$$

$$\therefore \bar{P}(n) \equiv 2 (r_1 + 1) \dots (r_k + 1) - 2 \cdot 3 (r_1 + 1) \dots (r_k + 1) \pmod{8}$$

[since  $\sigma(n) \equiv 3 (r_1 + 1) \dots (r_k + 1) \pmod{4}$ , where  $a$  is not zero]

$$\bar{P}(n) \equiv -4 (r_1 + 1) \dots (r_k + 1) \pmod{8}$$

$$\equiv 4 (r_1 + 1) \dots (r_k + 1) \pmod{8}$$

$$\equiv 4 \pmod{8}.$$

[since  $r_i$ 's and  $s_j$ 's are even integers so,  $(r_1 + 1) \dots (r_k + 1) \equiv 1 \pmod{8}$ ] [Fortin et al

(2005)]

$\therefore \bar{P}(n) \equiv 4 \pmod{8}$ , when  $n$  is a double of a square. Hence the Theorem 3 .

[Numerical example 3: If  $n$  is a double of a square. We get;  $\bar{P}(2) = 4$ ,  $\bar{P}(8) = 100$ , ...

$$\therefore \bar{P}(2) = 4 \equiv 4 \pmod{8}, \bar{P}(8) = 100 \equiv 4 \pmod{8}, \dots$$

We can conclude that  $\bar{P}(n) \equiv 4 \pmod{8}$ , for such  $n$ .]

Suppose that  $a$  is even. Then  $n$  is a square of an even integer.

From (1) we get;  $\bar{P}(n) \equiv 2 (r_1 + 1) \dots (r_k + 1) - 2 \sigma(n) + 4 \pmod{8}$



[since  $d_{1,4}(n) - d_{3,4}(n) = (r_1 + 1) \dots (r_k + 1)$  where  $r_i$ 's and  $s_j$ 's are even integers,  $\chi(n) = 1$ , where  $n$  is a square of an integer and  $d(n) \equiv 1 \pmod{8}$ ].

$$\text{or } \bar{P}(n) \equiv 2 (r_1 + 1) \dots (r_k + 1) + 2 - 2 \cdot 3 (r_1 + 1) \dots (r_k + 1) \pmod{8}$$

$$[\text{since } \sigma(n) \equiv 3 (r_1 + 1) \dots (r_k + 1) \pmod{4}, \text{ where } a \neq 0]$$

$$\text{or } \bar{P}(n) \equiv -4 (r_1 + 1) \dots (r_k + 1) + 2 \pmod{8}$$

$$\equiv 4 (r_1 + 1) \dots (r_k + 1) + 2 \pmod{8}$$

$$\equiv 4 \cdot 1 + 2 \pmod{8}$$

$$[\text{since } r_i \text{'s and } s_j \text{'s are even integers so } (r_1 + 1) \dots (r_k + 1) \equiv 1 \pmod{8}].$$

$\therefore \bar{P}(n) \equiv 6 \pmod{8}$ , when  $n$  is a square of an even integer. Hence the Theorem 4 .

[Numerical example 4: If  $n$  is a square of an even integer. We get;  $\bar{P}(4) = 14, \dots$

$$\therefore \bar{P}(4) = 14 \equiv 6 \pmod{8}, \dots$$

We can conclude that  $\bar{P}(n) \equiv 6 \pmod{8}$ , for such  $n$ .]

## 5. CONCLUSION

In this study we have analyzed various relations  $\bar{P}(n) = \sum_k 2^k p_k(n)$ ,  $\bar{P}(3n + 2) \equiv 0 \pmod{4}$ ,

$$\bar{P}(4n + 3) \equiv 0 \pmod{8}, \bar{P}(8n + 7) \equiv 0 \pmod{64},$$

$$d_{1,4}(n) - d_{3,4}(n) = \begin{cases} (r_1 + 1) \dots (r_k + 1), & \text{if } s_i \text{'s are even integers,} \\ 0, & \text{otherwise} \end{cases},$$

$$\bar{P}(n) \equiv 2d_{1,4}(n) - 2d_{3,4}(n) - 2\chi(n) - 2\sigma(n) + 4d(n) \pmod{8},$$

$$\sigma(n) = (2^{a+1} - 1) \prod_i \left( \sum_{m=0}^{r_i} p_i^m \right) \prod_j \left( \sum_{m=0}^{s_j} q_j^m \right), \text{ and } \sigma(n) \equiv \begin{cases} (r_1 + 1) \dots (r_k + 1) \pmod{4} & \text{if } a = 0 \\ 3(r_1 + 1) \dots (r_k + 1) \pmod{4}, & \text{otherwise,} \end{cases}$$

Respectively with the help of numerical examples. We have verified the four Theorems about overpartitions modulo 8 with numerical examples.

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