Fuzzy approach with linear and some non-linear membership functions for solving multi-objective assignment problems

Kagade K. L. and Bajaj V. H. *  
*Department of Statistics, Dr. B. A. M. University, Aurangabad (M.S.)-431004, India  
E-mail: vhbajaj@gmail.com, kailaskagade@gmail.com

Abstract- Assignment Problem (A.P.) originates from the classical problems where the objective is to find the optimum assignment of a number of jobs (tasks) to an equal number of machines (or workers) at a minimum cost or minimum time. The multi-objective assignment problem refers to a special class of vector minimum linear programming problems. In this paper, we use a special type of linear and non-linear membership functions to solve the multi-objective assignment problem. It gives an optimal compromise solution. The result obtained by using a linear membership function has been compared with the solution obtained by using non-linear membership functions. Numerical example has been provided to illustrate the solution procedure.

Keywords- Assignment Problem, Multi-criteria decision making, Linear membership function, Non-linear membership function

Introduction

The Assignment Problem (A.P.) is one of the most-studied, well-known and important problem in mathematical programming in which our objective is to assign a number of jobs (tasks) to an equal number of machines (workers) so as to minimize the total assignment cost or to minimize the total consumed time for execution of all the jobs (tasks). Hence assignment problem can be viewed as a balanced transportation problem, in which all supplies and demands equal to 1, and the number of rows and columns in the matrix are identical. Hence, Ravindran et al [6] can be used the transportation simplex method to solve the assignment problems. However, as an assignment problem is highly degenerate it will be frustrating or inefficient and not recommended to attempt to solve it by simplex method. Another technique called Hungarian method is commonly employed to solve the minimizing assignment problem by Ravindran et al [7]. Geetha et al [3] first expressed the cost-time minimizing assignment as the multicriteria problem. Bit et al[1] applied the fuzzy programming technique with linear membership function to solve the multi-objective transportation problem. Tsai et al [10] provided a solution for balanced multi-objective decision making problem associated with cost, time and quality by fuzzy concept. The Linear Interactive and Discrete Optimization (LINDO) Schrage [8], General Interactive Optimizer (GINO) Liebman[5] and TORA packages Taha[9] as well as many other commercial and academic packages are useful to find the solution of the assignment problem. Zadeh [12] first introduced the concept of fuzzy set theory. Then, Zimmermann [13] first applied suitable membership functions to solve linear programming problem with several objective functions. He showed that solutions obtained by fuzzy linear programming are always efficient. Verma et al [11] used the fuzzy programming technique with some non-linear (hyperbolic and exponential) membership functions to solve a multi-objective transportation problem. Dhamra et al [2] defined other types of the non-linear membership functions and applied them to an optimal design problem. In the multi-objective assignment problem, only the objectives are considered as fuzzy. We apply the fuzzy approach with linear and some non-linear membership functions to solve a multi-objective assignment problem as a vector minimum problem.

Assumptions and notations

The following assumptions and notations are used in developing the proposed model:

There are n jobs (tasks) in a factory and the factory has n machines (workers) to process the jobs (tasks).

Each job can be associated with one and only one machine.

Penalty $C_{ij} \geq 0$ be the execution cost, time etc which is incurred when a job $i (i = 1, 2, \ldots, n)$ is processed by the machine $j (j = 1, 2, \ldots, n)$.

The crisp number $X_{ij}$ denotes that the $i^{th}$ job is assigned to the $j^{th}$ machine.

Each machine can perform each job but with varying degree of efficiency.

Mathematical Formulation

A Multi-objective assignment problem may be stated mathematically as:

\[ \text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}, \quad k = 1, 2, \ldots, K \]

subject to

\[ \sum_{j=1}^{n} X_{ij} = 1, \quad i = 1, 2, \ldots, n \]  
\[ \sum_{i=1}^{n} X_{ij} = 1, \quad j = 1, 2, \ldots, n \]

and

\[ X_{ij} = \begin{cases} 1, & \text{if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, & \text{if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases} \]  

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The constraint (2) ensures that only one job is assigned to one machine while the constraint (3) ensures that only one machine should be assigned to one job. And the subscript on $Z_k$ and superscript on $c_{ij}^k$ denote the kth objective function as follows:

Case (i) A linear membership function for the k-th objective function is defined by $\mu_k(X)$ and shown in Fig. (1).

$$
\mu_k(x) = \begin{cases} 1, & \text{if } Z_k \leq L_k \\ \frac{Z_k - L_k}{U_k - L_k}, & \text{if } L_k < Z_k < U_k \\ 0, & \text{if } Z_k \geq U_k \end{cases}
$$

Case (ii) An hyperbolic membership function for the k-th objective function is defined by $\mu^H_k(x)$

$$
\mu^H_k(x) = \begin{cases} 1, & \text{if } Z_k \leq L_k \\ \frac{Z_k - L_k}{U_k - L_k}, & \text{if } L_k < Z_k < U_k \\ 0, & \text{if } Z_k \geq U_k \end{cases}
$$

Case (iii) An exponential membership function for the k-th objective function is defined by $\mu^E_k(x)$

$$
\mu^E_k(x) = \begin{cases} 1, & \text{if } Z_k \leq L_k \\ \frac{Z_k - L_k}{U_k - L_k}, & \text{if } L_k < Z_k < U_k \\ 0, & \text{if } Z_k \geq U_k \end{cases}
$$

Where, $a_k = \frac{6}{(U_k - L_k)}$

$S$ is a non-zero parameter, prescribed by the decision maker.

Step 5: From step 4, we can find an equivalent crisp model for the initial fuzzy model as follows:

If we will use the linear membership function as defined in (5) then an equivalent crisp model for the fuzzy model can be formulated as:

Maximize $\lambda$ subject to

$$
\lambda \leq \frac{u_k Z_k(X)}{U_k - L_k}, \quad k=1,2,...,K
$$

$\prod_{i=1}^{n} X_{ij} = 1,$  \hspace{1cm} $j=1,2,...,n; \quad \prod_{j=1}^{n} X_{ij} = 1,$  \hspace{1cm} $i=1,2,...,n; \quad \lambda \geq 0$

$X_{ij} = \begin{cases} 1, & \text{if the i}^{\text{th}} \text{ job is assigned to the j}^{\text{th}} \text{machine} \\ 0, & \text{if the i}^{\text{th}} \text{ job is not assigned to the j}^{\text{th}} \text{machine} \end{cases}$
The above problem can be further simplified as:
Maximize $\lambda$
subject to
$$Z_k(X) = \frac{1}{2}(U_k + L_k) \leq \frac{1}{2}(U_k + L_k), \quad k = 1,2,...,K$$
$$\sum_{i=1}^{n} x_{ij} \leq 1, \quad i = 1,2,...,n; \quad \sum_{j=1}^{m} x_{ij} = 1, \quad i = 1,2,...,n; \quad \lambda \geq 0$$
$$x_{ij} = \begin{cases} 1, & \text{if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, & \text{if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases}$$

If we use a hyperbolic membership function as defined in (6), an equivalent crisp model can be formulated as:
Maximize $\lambda$
subject to
$$Z_k(X) + \frac{1}{2}(U_k + L_k) \leq \frac{1}{2}(U_k + L_k), \quad k = 1,2,...,K$$
$$\sum_{i=1}^{n} x_{ij} = 1, \quad i = 1,2,...,n; \quad \sum_{j=1}^{m} x_{ij} = 1, \quad i = 1,2,...,n; \quad \lambda \geq 0$$
$$x_{ij} = \begin{cases} 1, & \text{if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, & \text{if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases}$$

The above problem can be further simplified as:
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$$\sum_{i=1}^{n} x_{ij} = 1, \quad i = 1,2,...,n; \quad \sum_{j=1}^{m} x_{ij} = 1, \quad i = 1,2,...,n; \quad \lambda \geq 0$$
$$x_{ij} = \begin{cases} 1, & \text{if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, & \text{if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases}$$

Step 6: Solve the crisp model by an appropriate mathematical programming algorithm.

The solution obtained in step 6 will be the optimal compromise solution of the Multi-objective assignment problem.

### Numerical Example

Minimize $Z_1 = \max (42, 28) = 42$, for the objective $Z_1$

Minimize $Z_2 = \max (29, 38) = 38$, for the objective $Z_2$

Minimize $Z_3 = \max (12, 31) = 31$

Subject to
$$\sum_{i=1}^{n} x_{ij} = 1, \quad i = 1,2,...,n; \quad \sum_{j=1}^{m} x_{ij} = 1, \quad i = 1,2,...,n; \quad \lambda \geq 0$$
$$x_{ij} = \begin{cases} 1, & \text{if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, & \text{if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases}$$

For the objective $Z_1$, we find the optimal solution as
$$X(1) = \begin{bmatrix} 12 & 1, & 23 & 1, & 31 & 1, \ & & & & & \\ \end{bmatrix}$$
and $Z_1 = 29$

For the objective $Z_2$, we find the optimal solution as
$$X(2) = \begin{bmatrix} 13 & 1, & 21 & 1, & 32 & 1, \ & & & & & \\ \end{bmatrix}$$
and $Z_2 = 28$

We can write the payoff matrix as
$$X(1) = \begin{bmatrix} 2 & 9 & 38 \\ 4 & 2 & 8 \end{bmatrix}$$

From the payoff matrix we find the upper bound and lower bound
$$U_1 = \max (29, 38) = 38, \quad U_2 = \max (42, 28) = 42, \quad L_1 = 29, \quad L_2 = 28, \quad d_1 = 9, \quad d_2 = 14$$

If we use the linear membership function as defined in(5), an equivalent crisp model can be formulated as:
Maximize $\lambda$
subject to
$$-Z(X) + k X_{nn+1} \leq \frac{1}{2}(U_k + L_k), \quad k = 1,2,...,K$$
$$\sum_{i=1}^{n} x_{ij} = 1, \quad i = 1,2,...,n; \quad \sum_{j=1}^{m} x_{ij} = 1, \quad i = 1,2,...,n; \quad \lambda \geq 0$$
$$x_{ij} = \begin{cases} 1, & \text{if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, & \text{if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases}$$

The problem is solved by the linear interactive and discrete optimization (LINDO) software. The optimal solution is presented as follows:
$$X^* = \begin{bmatrix} 1, & 1, & 1, & 1, & 1, & 1, \ & & & & & \\ \end{bmatrix}$$
and all $X_{ij}$’s are zeros

$Z_1^* = 30, \quad Z_2^* = 37$ and $\lambda = 0.58$

If we use the hyperbolic membership function as defined in (6), an equivalent crisp model can be formulated as:
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Minimize $x_0$

Subject to

$\sum_{j=1}^{3} x_{ij} = 1, \quad j=1, 2, 3$  
$\sum_{i=1}^{3} x_{ij} = 1, \quad i=1, 2, 3$  
$x_{ij} \geq 0, \quad i=1, 2, 3; \quad j=1, 2, 3$

The problem is solved by the linear interactive and discrete optimization (LINDO) software. The optimal solution is presented as follows:

$X^* = \begin{cases} 
X_{12} = 1, \quad X_{21} = 1, \quad X_{33} = 1, \\
\text{and rest all } X_{ij} \text{'s are zeros}
\end{cases}$

But here,

$X_{n+1} = \tanh^{-1}(2\lambda - 1)\tanh(0.4818653) = 2\lambda - 1$

Therefore

$Z_1^* = 30, \quad Z_2^* = 37$ and $\lambda = 0.50$

However, if we use exponential membership function as defined in (7) with the parameter $S=1$, an equivalent crisp model for the fuzzy model can be formulated as:

Maximize $\lambda$

Subject to

$exp\{(9X_{11} \cdot 8X_{12} \cdot 15X_{13} \cdot 12X_{21} \cdot 10X_{22} \cdot 12X_{23} \cdot 18X_{31} \cdot 9X_{32} \cdot 29X_{33})/9 - 63/202\} \geq 0.3678794$

The problem is solved by the General Interactive Optimizer (GINO) software. The optimal solution is presented as follows:

$X^* = \begin{cases} 
X_{12} = 1, \quad X_{21} = 1, \quad X_{33} = 1, \\
\text{and rest all } X_{ij} \text{'s are zeros}
\end{cases}$

$Z_1^* = 30, \quad Z_2^* = 37$ and $\lambda = 0.45$

**Conclusion**

In this paper, linear and non-linear membership functions have been used to solve the multi-objective assignment problem. If we use the hyperbolic membership function, then the crisp model becomes linear. The optimal compromise solution does not change significantly if we compare with the solution obtained by the linear membership function. However, if we use the exponential type membership function, with different values of $S$ (parameter) then the crisp model becomes non-linear and the optimal compromise solution does not change significantly, if we compare with the solution obtained by the linear membership function.

**References**