SIMILARITY ANALYSIS OF UNSTEADY THREE DIMENSIONAL BOUNDARY LAYERS OF A NON-NEWTONIAN MODEL

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ABSTRACT

In this study, three dimensional, unsteady, laminar boundary layer equations of a general model of non-Newtonian fluids are treated. In this model, the shear stresses are considered to be arbitrary functions of velocity gradients. A general boundary value problem modeling the flow over a moving surface with suction or injection is considered. Using Similarity Analysis, we showed that equations admit scaling transformation for the arbitrary shear stress case. The specific forms of the stress functions where richer scaling symmetries exist are derived. We reduce the three-independent-variable partial differential system to two-independent-variable partial differential system. Using further translation symmetries of the outcome equations, the boundary value problem is successfully reduced to an ordinary differential system.

Key Words : Non-Newtonian fluids, Boundary layer

NON-NEWTONYEN BİR MODEL'İN ÜÇ BOYUTLU SINIR TABAKASI DENKLEMLERİNİN BENZERLİK ANALİZİ

ÖZET


Anahtar Kelimeler : Non-Newtonyan akışkanlar, Sınır tabakası

1. INTRODUCTION

We treat the unsteady boundary layer equations of a general non-Newtonian fluid model first proposed by Hansen and Na (1968). In their model, they take the shear stress as an arbitrary function of the velocity gradient. The model is a generalization of the visco-inelastic behaviour observed in several fluids including Newtonian, Power-Law, Williamson, Prandtl, Powel-Eyring, Eyring, Ellis and Reiner-Philipoff fluids. Hansen and Na (1968) presented a similarity solution for the steady two dimensional case using scaling transformation. Timol and Kalthia (1986) extended the analysis to three dimensions using scaling and spiral group transformations. Pakdemirli (1994) retreated the analysis of references Hansen and Na, (1968) and Timol and Kalthia, (1986) showed that richer similarities exits for some specific forms of the stress function. Recently, Pakdemirli et al. (1996)
used exterior differential forms to determine the general symmetries of the two dimensional steady-state equations of the model. Yürüşoy (1996) and Yürüşoy, Pakdemirli, (1996) calculated the symmetries of the unsteady two-dimensional boundary layer equations by applying Lie Group analysis.

In reference (Yürüşoy, 1996), the classical boundary layer problem, flow with suction or injection and flow over a stretching sheet cases are investigated whereas in reference (Yürüşoy and Pakdemirli, 1996) the combined effects of moving surface with suction or injection are treated. For the boundary value problems of reference (Yürüşoy, 1996), reduction for the partial differential system from three independent variables to two independent variables is possible whereas further reduction to ordinary differential equations is impossible. However, in reference (Yürüşoy and Pakdemirli 1996), it is shown that the three independent variable partial differential system corresponding to moving surface with suction or injection can be reduced to ordinary differential system by successive application of Lie Groups. In this work, we treat the three dimensional unsteady boundary layer equations. The boundary value problem is the same as in Yürüşoy and Pakdemirli (1996) with a generalization to three dimensions. We showed that equations admit scaling transformation for the arbitrary shear stress case. The specific forms of the stress functions where richer scaling symmetries exist are derived. By assuming all flow quantities to be independent of z coordinate, we reduce the three-independent-variable partial differential system to two-independent-variable partial differential system. Using further translation symmetries of the resulting equations, the boundary value problem is successfully reduced to an ordinary differential system.

2. EQUATIONS OF MOTION

The three dimensional incompressible, laminar, unsteady, boundary layer equations have the following form,

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{\partial \tau_{xy}}{\partial y} \]  

\[ + \frac{\partial u}{\partial t} + U \frac{\partial U}{\partial x} + W \frac{\partial U}{\partial z} \]  

\[ + \frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} + W \frac{\partial w}{\partial z} = \frac{\partial \tau_{yz}}{\partial y} \]  

\[ + \frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} + W \frac{\partial w}{\partial z} \]  

\[ F(\tau_{xy}, \frac{\partial u}{\partial y}, \frac{\partial w}{\partial y}) = 0 \]  

\[ G(\tau_{yz}, \frac{\partial u}{\partial y}, \frac{\partial w}{\partial y}) = 0 \]  

where the shear stresses and velocity gradients are implicitly related through the arbitrary continuous functions F and G. Note that the components of velocity gradients which are largest inside the boundary layer are taken into consideration. U and W denote the x and z components of velocities outside the boundary layer.

3. SCALING SYMMETRIES

In this section, we apply scaling transformation to equations (1)-(5). Two cases are of practical importance: 1) Arbitrary shear stress, 2) Specific forms of stresses where richer symmetries exits. We scale all the independent and dependent variables as follows,

\[ \tilde{x} = \lambda^a x, \quad \tilde{y} = \lambda^b y, \quad \tilde{z} = \lambda^c z, \quad \tilde{t} = \lambda^d t, \quad \tilde{u} = \lambda^e u, \]  

\[ \tilde{v} = \lambda^f v, \quad \tilde{w} = \lambda^g w \]  

\[ \tilde{U} = \lambda^d U, \quad \tilde{W} = \lambda^d W, \quad \tilde{\tau}_{xy} = \lambda^d \tau_{xy} \]  

Substituting the new variables defined in (6) into (1)-(5), requiring that the new system of equations have equivalent form with the old system results in the invariance conditions

\[ b + e - a - f = 0, \quad c + e - a - g = 0, \quad a - e - d = 0, \quad b - d - f = 0 \]  

\[ c - d - g = 0, \quad b + e - d - h = 0, \quad c - e - a = 0 \]  

\[ c + e - d - i - j = 0, \quad b + g - d - k = 0, \quad g - j = 0, \]  

\[ a + g - d - i - j = 0, \quad c + g - d - 2j = 0 \]  

\[ F(\lambda^{-h} \tilde{\tau}_{xy}, \lambda^{b-e} \frac{\partial \tilde{u}}{\partial y}, \lambda^{b-e} \frac{\partial \tilde{w}}{\partial y}) = F(\tau_{xy}, \frac{\partial u}{\partial y}, \frac{\partial w}{\partial y}) \]  

(8)
\[ G(\lambda^{-k} \tau_{yz}, \lambda^{-l} \tau_{xy}, \lambda^{-m} \tau_{x}) = G(\tau_{yz}, \tau_{xy}, \tau_{x}) \]  
(9)

### 3. 1. Arbitrary Shear Stress

Requiring \( F \) and \( G \) to remain arbitrary under the transformation yields

\[
\begin{align*}
\alpha &= d + e, \quad b = d + f, \quad c = d + g, \quad h = f + e, \quad i = e, \\
\beta &= j = g, \quad k = f + g
\end{align*}
\]

Substituting the new variables (13) and (15) into (1)-(5) and (14) and remembering that all flow quantities are independent of \( z \) coordinate, we finally obtain a partial differential system with two independent variables

\[
\begin{align*}
P_{\eta} + Q_{\eta} &= 0 \\
\frac{1}{2} P - \frac{3}{2} \tau_{xy} P_{\xi} - \frac{1}{2} \eta P_{\eta} + PP_{\eta} + QQ_{\eta} &= (\tau_{xy}) \eta \\
\frac{1}{2} U - \frac{3}{2} \tau_{xy} U_{\xi} + UU_{\xi} &= 0
\end{align*}
\]

### 3. 2. Specific Forms of Stresses

If we require that \( F \) and \( G \) functions in (8) and (9) possess scaling properties, then we obtain some special forms of the functions. For those special forms, obviously the scaling symmetries would be enriched. To manage this, we differentiate (8) and (9) with respect to \( \lambda \), return to original variables, solve the outcoming first order partial differential system, solve (7) and finally obtain

\[
\begin{align*}
P (\xi,0) &= A (\xi), \quad Q (\xi,0) = \pm V (\xi), \quad R (\xi,0) = B (\xi), \\
P (\xi,\infty) &= U (\xi), \quad R (\xi,\infty) = W (\xi)
\end{align*}
\]

The boundary conditions for the equations of motion

\[
\begin{align*}
u (x,0,t) &= A(x,t), \quad V (x,0,t) = \pm V (x,t) \quad W (x,0,t) = B (x,t), \\
u (x,\infty,t) &= U (x,t), \quad V (x,\infty,t) = W (x,t)
\end{align*}
\]

The boundary conditions imply that the surface is moving and there is suction or injection through the surface. Requiring that the functions \( A(x,t), V(x,t) \) and \( B(x,t) \) possess scaling properties yield

\[
\begin{align*}
A (x,t) &= t^{\frac{1}{2}} A (\xi), \quad V (x,t) = t^{\frac{1}{2}} V (\xi), \\
B (x,t) &= t^{\frac{1}{2}} B (\xi)
\end{align*}
\]
Group scaling transformation (see eq. (11)). Therefore symmetries are richer in this case compared to the arbitrary shear stress case. Note that Newtonian and Power-Law fluids obey the general form given in (23) and (24). Applications of Newtonian and Power-Law fluids for the symmetries given in (22)-(24) would be similar to those given in reference (Pakdemirli, 1994) with the exception that shear stresses are explicit functions of velocity gradients in Pakdemirli (1994) whereas in our case, they are implicit functions.

4. TRANSLATION SYMMETRIES OF REDUCED SYSTEM

In this section, we treat equations (16)-(21) obtained for the arbitrary shear stress case. From the results given in reference (Yürüsoy and Pakdemirli 1996), we expect the reduced system (16)-(21) to possess translation symmetries only. We therefore write

\[ \hat{\xi} = \xi + \alpha \xi \], \( \hat{\eta} = \eta + \beta \eta \), \( \hat{P} = P + \alpha P \),
\[ \hat{Q} = Q + \alpha Q \], \( \hat{R} = R + \alpha R \), \( \hat{\tau}_{xy} = \tau_{xy} + \alpha \tau_{xy} \),
\[ \hat{U} = U + \alpha U \], \( \hat{W} = W + \alpha W \), \( \hat{\tau}_{yz} = \tau_{yz} + \alpha \tau_{yz} \),
\[ \hat{A} = A + \alpha A \], \( \hat{V} = V + \alpha V \), \( \hat{B} = B + \alpha B \)

Substituting (25) into (16)-(21) and remembering that F and G are arbitrary, we obtain the following invariance conditions

\[ g = c, \quad \xi = \frac{3}{2} a, \quad \eta = \frac{3}{2} b, \quad h = e, \quad f = 0, \quad i = 0, \quad b = 0, \quad j = c, \quad k = d, \quad l = e \] (26)

The above equations can be represented in terms of two arbitrary parameters m and p as follows

\[ a = 2m, \quad b = 0, \quad c = 3m, \quad d = 0, \quad e = p, \quad f = 0, \quad g = 3m, \quad h = p, \quad i = 0, \quad j = 3m, \quad k = 0, \quad l = p \] (27)

Choosing m = 1 and p = 3, we write the differential system for the similarity variables and functions

\[ \frac{d\xi}{2} = \frac{\eta}{3}, \quad \frac{dP}{3} = \frac{dQ}{3} = \frac{dR}{3} = \frac{d\tau_{xy}}{0} = \frac{dU}{3} = \frac{dW}{3} = \frac{d\tau_{yz}}{0} = \frac{dA}{3} = \frac{dV}{3} = \frac{dB}{3} \] (28)

Solving (28), we find

\[ \mu = \eta, \quad P = \frac{3}{2} \xi + L(\mu), \quad Q = M(\mu), \quad R = \frac{3}{2} \xi + N(\mu), \quad U = \frac{3}{2} \xi + C_1 \] (29)
\[ W = \frac{3}{2} \xi + C_2, \quad A = \frac{3}{2} \xi + C_3, \quad V = C_4, \quad B = \frac{3}{2} \xi + C_5 \]

where \( \tau_{xy} \) and \( \tau_{yz} \) are absolute invariants again. Substituting (29) into (16)-(21), we have

\[ M' + \frac{3}{2} = 0 \] (30)
\[ 2L + L'(M - \frac{1}{2} \mu) = (\tau_{xy}) \mu + 2C_1 \] (31)
\[ \frac{1}{2} N + \frac{3}{2} L + N'(M - \frac{1}{2} \mu) = (\tau_{yz}) \mu + \frac{1}{2} C_2 + \frac{3}{2} C_1 \] (32)
\[ F(\tau_{xy}, L', N') = 0 \] (33)
\[ G(\tau_{yz}, L', N') = 0 \] (34)
\[ L(0) = C_3, \quad M(0) = \pm C_4, \quad N(0) = C_5, \quad L(\infty) = C_1, \quad N(\infty) = C_2 \] (35)

We therefore successfully reduced the partial differential system of three independent variables to an ordinary differential system by applying first scaling and then translation symmetries. A closed form solution of ordinary differential system (30)-(35) cannot be achieved unless we specify F and G. Even for specific F and G, however, a numerical treatment of equations might be inevitable.

5. REFERENCES


