Pebbling Number for Jahangir Graph $J_{2,m} (3 \leq m \leq 7)$

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Abstract: Given a configuration of pebbles on the vertices of a connected graph $G$, a pebbling move (or pebbling step) is defined as the removal of two pebbles off a vertex and placing one pebble on an adjacent vertex. The pebbling number, $f(G)$, of a graph $G$ is the least number $m$ such that, however $m$ pebbles are placed on the vertices of $G$, we can move a pebble to any vertex by a sequence of pebbling moves. In this paper, we determine $f(G)$ for Jahangir graph $J_{2,m} (3 \leq m \leq 7)$.

Key words: pebbling, Jahangir graph.

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1 Introduction

One recent development in graph theory, suggested by Lagarias and Saks, called pebbling, has been the subject of much research. It was first introduced into the literature by Chung [1], and has been developed by many others including Hulbert, who published a survey of pebbling results in [3]. There have been many developments since Hulbert’s survey appeared.

Given a graph $G$, distribute $k$ pebbles (indistinguishable markers) on its vertices in some configuration $C$. Specifically, a configuration on a graph $G$ is a function from $V(G)$ to $\mathbb{N} \cup \{0\}$ representing an arrangement of pebbles on $G$. For our purposes, we will always assume that $G$ is connected.

A pebbling move (or pebbling step) is defined as the removal of two pebbles from some vertex and the placement of one of these pebbles on an adjacent vertex. Define the pebbling number, $f(G)$, to be the minimum number of pebbles such that regardless of their initial configuration, it is possible to move to any root vertex $v$, a pebble by a sequence of pebbling moves. Implicit in this definition is the fact that if after moving to vertex $v$ one desires to move to another root vertex, the pebbles reset to their original configuration.

**Fact 1.1** [5,6]. For any vertex $v$ of a graph $G$, $f(v,G) \geq n$ where $n=|V(G)|$.

**Proof.** Consider the configuration $C:V-\{v\} \rightarrow \mathbb{N}$ defined by $C(w)=1$ for all $w \in V-\{v\}$. Then the size $|C|$ of the configuration is $n-1$. In this configuration we cannot move a pebble to $v$. Hence $f(v,G) \geq |V(G)|$.

**Fact 1.2** [5]. The pebbling number of a graph $G$ satisfies $f(G) \geq \max \{2^{\text{diam}(G)} \cdot |V(G)| \}$.

**Proof.** Let $w \in V(G)$ be a vertex at a distance $\text{diam}(G)$ from the target vertex $v$. Place $2^{\text{diam}(G)}-1$ pebbles at $w$. Clearly we cannot move any pebble to $v$. Thus $f(G) \geq \max \{2^{\text{diam}(G)} \cdot |V(G)| \}$.
There are few other interesting results in the pebbling number of graphs. Hulbert [3] has written an excellent survey article on pebbling. We earnestly request the interested readers to refer to it for further study.

We also request the readers to read [2] in which Moews has studied the pebbling number of product of trees.

We now proceed to determine the pebbling number for \( J_{2,m} \) (3 ≤ m ≤ 7).

2 Pebbling Number of Jahangir Graph \( J_{2, m} \) (3 ≤ m ≤ 7)

Definition 2.1 [4] Jahangir graph \( J_{h, m} \) for m ≥ 3 is a graph on \( nm + 1 \) vertices, that is, a graph consisting of a cycle \( C_{nm} \) with one additional vertex which is adjacent to \( m \) vertices of \( C_{nm} \) at distance \( n \) to each other on \( C_{nm} \).

Example: Figure 1 shows Jahangir graph \( J_{2, 8} \). The figure \( J_{2,8} \), appears on Jahangir’s tomb in his mausoleum. It lies in 5 kilometer north-west of Lahore, Pakistan, across the River Ravi.

\[ \text{Figure 1: } J_{2,8} \]

Remark 2.2 Let \( v_{2m + 1} \) be the label of the center vertex and \( v_1, v_2, \ldots, v_{2m} \) be the label of the vertices that are incident clockwise on cycle \( C_{2m} \) so that \( \deg(v_1) = 3 \).

Theorem 2.3 For the Jahangir graph \( J_{2, 3} \), \( f(J_{2,3}) = 8 \).
Proof Put seven pebbles at $v_4$. Clearly we cannot move a pebble to $v_1$, since $d(v_4, v_1) = 3$. Thus $f(J_{2,3}) \geq 8$.

Now, consider a distribution of eight pebbles on the vertices of $J_{2,3}$. Consider the sets $S_1 = \{v_1, v_3, v_5\}$ and $S_2 = \{v_2, v_4, v_6\}$.

Case (i) Suppose we cannot move a pebble to the vertex $v_7$.

The vertices of $S_1$ must contain at most one pebble each. The vertices of $S_2$ contain at most three pebbles each. If any vertex of $S_1$ contains a pebble, then the neighbors of that vertex contain at most one pebble each. Otherwise, a pebble can be moved to $v_7$. Also, note that, if a vertex of $S_2$ contains two or three pebbles then the neighbors of that vertex contain zero pebbles and also no other vertex of $S_2$ contain two or three pebbles. Otherwise, we can move a pebble to $v_7$. So, in any such distributions, the vertices of $J_{2,3}$ contain at most six pebbles so that a pebble could not be moved to $v_7$—a contradiction to the total number of pebbles placed over the vertices of $J_{2,3}$. Thus, we can always move a pebble to $v_7$. So, our assumption (Case(i)) is wrong.

Case (ii) Suppose we cannot move a pebble to a vertex of $S_1$. Without loss of generality, let $v_1$ be the target vertex.

Clearly, the neighbors $v_2, v_6, v_7$ of $v_1$ contain at most one pebble each. The vertices $v_3, v_5$ contain at most three pebbles each and the vertex $v_4$ contains at most seven pebbles. Also, note that, both $v_3$ and $v_5$ cannot contain two or three pebbles each.
(otherwise, a pebble can be moved to $v_1$ through $v_7$) and if either $v_3$ or $v_5$ contains two or three pebbles then $v_4$ contains at most three pebbles.

Suppose $v_7$ has a pebble on it, then the vertices $v_3$ and $v_5$ contain at most one pebble each. This implies that, the vertex $v_4$ contains at least three pebbles on it. If any one of the vertex $v_3$ or $v_5$ has a pebble on it, then we can easily move a pebble to $v_1$. Otherwise, $v_4$ contains at least four pebbles on it and so we can move a pebble to $v_1$ through $v_7$. So, assume that $v_7$ has zero pebbles on it.

Suppose $v_2$ has a pebble on it. Then, the path $v_4v_5$ contains at least five pebbles. Clearly, we can move a pebble to $v_1$. So, assume that $v_2$ has zero pebbles on it. In a similar way, we may assume that, $v_6$ has zero pebbles on it.

Suppose $v_3$ has two or three pebbles on it. Then, $v_4$ contains at least four pebbles on it and so we can move a pebble to $v_1$ easily. So, assume $v_3$ contains at most one pebble on it. In a similar way, we may assume that, $v_5$ has at most one pebble on it. Then, $v_4$ contains at least six pebbles on it. If any one of the vertices $v_3$ or $v_5$ contains a pebble then we can move a pebble to $v_1$. Otherwise, $v_4$ contains eight pebbles on it and hence we can move a pebble to $v_1$. Thus, we can always move a pebble to $v_1$. So, our assumption (Case(ii)) is wrong.

**Case (iii)** Suppose we cannot move a pebble to a vertex of $S_2$. Without loss of generality, let $v_2$ be the target vertex.

The neighbors $v_1$, and $v_3$ of $v_2$ contain at most one pebble each. The vertices $v_4$, $v_6$, and $v_7$ contain at most three pebbles each and the vertex $v_5$ contains at most seven pebbles. Suppose $v_3$ has a pebble on it. Then the path $v_5v_6$ contains at least four pebbles. If either $v_4$ or $v_7$ contains a pebble on it, clearly we can move a pebble to $v_2$. Otherwise, the path $v_5v_6$ contains at least five pebbles and so we can move a pebble to $v_2$ easily. So, assume that $v_3$ has zero pebbles on it. In a similar way, we may assume that, $v_1$ has zero pebbles on it.
Suppose $v_7$ has two or three pebbles on it. Then, clearly we can move a pebble to $v_2$, since $v_5$ contains at least four pebbles on it. So, assume that $v_7$ has at most one pebble on it. Suppose $v_4$ contains two or three pebbles on it. Then, the path $v_5v_6$ contains at least four pebbles and so we are done if $v_7$ contains a pebble. Otherwise, the path $v_5v_6$ contains at least five pebbles. Then, a pebble can easily be moved to $v_2$. So, assume that, $v_4$ contains at most one pebble. In a similar way, we may assume that, $v_6$ has at most one pebble on it. Then, $v_5$ contains at least five pebbles. If $v_7$ contains one pebble on it, then we are done if either $v_4$ or $v_6$ has one pebble on it. Otherwise, it is easy to move a pebble to $v_2$. So, assume that $v_7$ has zero pebbles on it. Then, $v_5$ contains at least six pebbles. In this case also it is easy to move a pebble to $v_2$. Thus, we can always move a pebble to $v_2$. So, our assumption (Case(iii)) is wrong. Thus, $f(J_{2,3}) \leq 8$.

Therefore, $f(J_{2,3}) = 8$.

**Theorem 2.4** For the Jahangir graph $J_{2,4}$, $f(J_{2,4}) = 16$. 

![J_{2,4} Graph](image)
Proof Put fifteen pebbles at \( v_8 \) then we cannot move a pebble to \( v_4 \), since \( d(v_4, v_8) = 4 \). Thus \( f(J_{2,4}) \geq 16 \).

Now, consider a distribution of sixteen pebbles on the vertices of \( J_{2,4} \). Consider the sets \( S_1 = \{v_1, v_3, v_5, v_7\} \) and \( S_2 = \{v_2, v_4, v_6, v_8\} \).

**Case (i)** Suppose we cannot move a pebble to the vertex \( v_9 \).

Clearly, the vertices of \( S_1 \) contain at most one pebble each and the vertices of \( S_2 \) contain at most three pebbles each. If any vertex of \( S_1 \) contains a pebble, then the neighbors of that vertex contain at most one pebble each. Otherwise, a pebble can be moved to \( v_9 \). Also, note that, if a vertex of \( S_2 \) contains two or three pebbles then the neighbors of that vertex contain zero pebbles and also the proceeding and succeeding vertex of that vertex in \( S_2 \) contain at most one pebble each. Otherwise, we can move a pebble to \( v_9 \). So, in any such distribution, the vertices of \( J_{2,4} \) contain at most eight pebbles so that a pebble could not be moved to \( v_9 \) - a contradiction to the total number of pebbles placed over the vertices of \( J_{2,4} \). Thus, we can always move a pebble to \( v_9 \). So, our assumption (case(i)) is wrong.

**Case (ii)** Suppose we cannot move a pebble to a vertex of \( S_1 \). Without loss of generality, let \( v_1 \) be the target vertex.

Clearly, the neighbors \( v_2, v_8 \), and \( v_9 \) of \( v_1 \) contain at most one pebble each. The vertices of \( S_1 - \{v_1\} \) contain at most three pebbles and the vertices of \( S_2 - \{v_2, v_8\} \) contain at most seven pebbles. Also, note that, no two vertices of \( S_1 - \{v_1\} \) contain two or three pebbles each and if a vertex of \( S_1 - \{v_1\} \) contain two or three pebbles then the neighbor of that vertex contain at most three pebbles. Also, both \( v_4 \) and \( v_6 \) cannot contain four or more (at most seven) pebbles each and if either \( v_4 \) or \( v_6 \) contains four or more pebbles then the neighbors of that vertex contain at most one pebble.
Suppose \( v_9 \) has a pebble on it. Clearly, the neighbors \( v_3, v_5, \) and \( v_7 \) of \( v_9 \) contain at most one pebble each. So the remaining ten pebbles are at \( v_4 \) and \( v_6 \). Thus either \( v_4 \) or \( v_6 \) receives at least five pebbles. This implies that, we can move a pebble to \( v_1 \) through \( v_9 \)-a contradiction to case(ii). So assume that \( v_9 \) has zero pebbles on it.

Suppose \( v_2 \) has a pebble on it. Clearly, the neighbor \( v_3 \) of \( v_2 \) contains at most one pebble. If \( v_3 \) contains a pebble, then the vertex \( v_4 \) contains at most one pebble. Otherwise, \( v_4 \) contains at most three pebbles. So the path \( v_4v_3v_2v_1v_8v_7 \) contains at most seven pebbles. Now, the path \( v_5v_6 \) contain at least nine pebbles. Clearly, \( v_6 \) contains at least six pebbles. If either \( v_5 \) or \( v_7 \) or \( v_8 \) contains one pebble then we can move a pebble to \( v_1 \). Otherwise, the vertex \( v_6 \) contains at least eight pebbles and so we get a contradiction to case(ii). So assume \( v_2 \) has zero pebbles on it. In a similar way, we may assume that \( v_8 \) has zero pebbles on it.

Suppose \( v_5 \) has two or three pebbles on it. Then the path \( v_4v_5v_6 \) contains at most six pebbles, otherwise we get a contradiction to case (ii). Thus either \( v_3 \) or \( v_7 \) contains at least five pebbles-a contradiction to case(ii). Thus, assume that \( v_5 \) has at most one pebble. The vertices \( v_3 \) and \( v_7 \) totally contain at most four pebbles. This implies that, the path \( v_4v_5v_6 \) contains at least twelve pebbles. Clearly we can move a pebble to \( v_1 \)-a contradiction to case(ii). Thus, assume that \( v_5 \) has zero pebbles on it. So, either the path \( v_3v_4 \) or \( v_6v_7 \) contains at least eight pebbles. Clearly, we can move a pebble to \( v_1 \)-a contradiction to case(ii). Thus, we can always move a pebble to \( v_9 \). So, our assumption (Case(ii)) is wrong.

**Case (iii)** Suppose we cannot move a pebble to a vertex of \( S_2 \). Without loss of generality, let \( v_2 \) be the target vertex.

As we discussed in case (i) and case (ii), we can give a simple argument which will conclude that our assumption (case(iii)) is wrong.
Thus, $f(J_{2,4}) \leq 16$.

Therefore, $f(J_{2,4}) = 16$.

**Theorem 2.5** For the Jahangir graph $J_{2,5}$, $f(J_{2,5}) = 18$.

**Proof** Put fifteen pebbles at $v_6$ and one pebble each at $v_8$ and $v_{10}$, and then we cannot move a pebble to $v_2$. Thus $f(J_{2,5}) \geq 18$.

Now, consider a distribution of eighteen pebbles on the vertices of $J_{2,5}$. Consider the sets $S_1 = \{v_1, v_3, v_5, v_7, v_9\}$ and $S_2 = \{v_2, v_4, v_6, v_8, v_{10}\}$.

**Case (i)** Suppose we cannot move a pebble to the vertex $v_{11}$.

The vertices of $S_1$ contain at most one pebble each and the vertices of $S_2$ contain at most three pebbles each. Also, note that, no two consecutive vertices of $S_2$ contain two or three pebbles each and if a vertex of $S_2$ contains two or three pebbles then the neighbors of that vertex contain zero pebbles. Also, if a vertex of $S_1$ contains one pebble then the neighbors of that vertex contain at most one pebble each. Thus, in any distribution, $J_{2,5}$ contains at most ten pebbles so that a pebble could not be moved to $v_{11}$ – a contradiction to the total number of pebbles placed over the vertices of $J_{2,5}$. So, our assumption is wrong.

**Case (ii)** Suppose we cannot move a pebble to a vertex of $S_1$. Without loss of generality, let $v_1$ be the target vertex.

Clearly, the neighbors $v_2$, $v_{10}$, and $v_{11}$ of $v_1$ contain at most one pebble each. The vertices of $S_1 - \{v_1\}$ contain at most three pebbles each and the vertices of $S_2 - \{v_2, v_{10}\}$ contain at most seven pebbles each. Also, note that, no two vertices of $S_1 - \{v_1\}$ contain two or three pebbles each and if a vertex of $S_1 - \{v_1\}$ contains two or three pebbles then the neighbors of that vertex contain at most three pebbles each. Also, no two vertices of $S_2 - \{v_2, v_{10}\}$ contain four or more pebbles each and if a vertex of $S_2 -$
Clearly, the neighbors $v_2$, $v_{10}$, and $v_{11}$ of $v_1$ contain at most one pebble each. Suppose $v_{11}$ contains one pebble on it. Clearly the vertices of $S_1$-{$v_1$} contain at most one pebble each. This implies that, the paths $v_3v_4v_5$ and $v_7v_8v_9$ contain at most three pebbles each. Thus, the vertex $v_6$ contains at least nine pebbles and so we can move a pebble to $v_1$ easily—a contradiction to case(ii). So assume that $v_{11}$ has zero pebbles on it.

Suppose $v_2$ has one pebble on it. The vertex $v_3$ contains at most one pebble and the vertex $v_4$ contains at most three pebbles. If the vertex $v_5$ contains two or three pebbles then either $v_6$ or $v_8$ contains at least four pebbles. So we get a contradiction to case(ii). Thus, assume that $v_5$ contains at most one pebble. In a similar way, we may assume that both $v_7$ and $v_9$ contain at most one pebble each. Thus, the vertices $v_6$ and $v_8$ contain at least ten pebbles totally. So, in any distribution of these ten pebbles on $v_6$ and $v_8$, we can always move a pebble to $v_1$ through $v_7$ and $v_{11}$. So assume that $v_2$ has zero pebbles on it. In a similar way, we may assume that $v_{10}$ has zero pebbles on it.

Suppose $v_3$ has two or three pebbles on it. Then clearly the total number of pebbles on the vertices $v_6$ and $v_8$ is at least nine. Thus, we can move a pebble to $v_1$ through $v_7$ and $v_{11}$. So, assume that $v_3$ contains at most one pebble. In a similar way, we may assume that $v_9$ contains at most one pebble. Suppose $v_5$ contains two or three pebbles on it then the vertices $v_6$ and $v_8$ totally contain at least nine pebbles, so we can move a pebble to $v_1$ through $v_{11}$—a contradiction to case(ii). So assume that $v_5$ contains at most one pebble. In a similar way, we may assume that $v_7$ contains at most one pebble. Now, any one of the vertices of $S_2$-{$v_2$, $v_{10}$} contains at least four pebbles, say $v_6$. If both neighbors of that vertex $v_6$ contain one pebble each then we can move a pebble to $v_1$ or if $v_4$ contains two or three pebbles and if a neighbor of $v_4$ contains one pebble then also we can move a pebble to $v_1$. Otherwise, $v_6$ contains more than eight pebbles—a contradiction to case(ii). Thus, in any distribution of eighteen pebbles on the vertices of $J_{2,5}$, we can always move a pebble to $v_1$. So, our assumption (case(ii)) is wrong.
Case (iii) Suppose we cannot move a pebble to a vertex of $S_2$. Without loss of generality, let $v_2$ be the target vertex.

Clearly, the neighbors $v_1$, and $v_3$ of $v_2$ contain at most one pebble each and the vertices $v_4$, $v_{10}$, and $v_{11}$ contain at most three pebbles each. The vertices of $S_1 - \{v_1, v_3\}$ contain at most seven pebbles each and the vertices of $S_2 - \{v_2, v_4, v_{10}\}$ contain at most fifteen pebbles each. Also, note that, no two vertices of $S_1 - \{v_1, v_3\}$ contain four or more pebbles each and if a vertex of $S_1 - \{v_1, v_3\}$ contains four or more pebbles then the neighbors of that vertex contain at most three pebbles each. Also, no two vertices of $S_2 - \{v_2, v_4, v_{10}\}$ contain eight or more pebbles each and if a vertex of $S_2 - \{v_2, v_4, v_{10}\}$ contains eight or more pebbles then the neighbors of that vertex contain at most three pebbles each.

Suppose $v_3$ has one pebble on it. Clearly, both $v_4$ and $v_{11}$ contain at most one pebble each. If $v_{11}$ has one pebble on it then the vertices of $S_1 - \{v_1, v_3\}$ contain at most one pebble each. This implies that, any one of the vertices $v_6$, $v_8$ or $v_{11}$ contains at least four pebbles. Thus we can move a pebble to $v_2$. So, assume that $v_{11}$ has zero pebbles on it. Then either one of the vertices of $S_2 - \{v_2, v_4\}$ receives at least four pebbles and a vertex of $S_1 - \{v_1, v_3\}$ receives at least two pebbles or any two vertices of $S_2 - \{v_2, v_4\}$ receive at least four pebbles each. So, we can always move a pebble to $v_2$-a contradiction to case(iii). So, assume that $v_3$ has zero pebbles on it. In a similar way, we may assume that $v_1$ has zero pebbles on it.

Suppose $v_{11}$ has two or three pebbles on it. Then both $v_4$ and $v_{10}$ contain at most one pebble each. This implies that, the total number of pebbles on the vertices $v_6$ and $v_8$ is at least nine. So, we can move a pebble to $v_2$-a contradiction to case(iii). Thus, assume $v_{11}$ has at most one pebble.
Suppose $v_4$ contains two or three pebbles on it. Then one of the vertices of $\{v_6, v_8\}$ contains at least four pebbles and if a vertex of $S_1 - \{v_1, v_3\}$ contains two or three pebbles then we can move a pebble to $v_2$. Otherwise, the total number of pebbles on the vertices $v_6$ and $v_8$ is at least eleven—a contradiction to case(iii). So, assume $v_4$ contains at most one pebble. In a similar way, we may assume that $v_{10}$ contains at most one pebble.

Suppose $v_7$ has four or more pebbles (at most seven pebbles) on it. Clearly, the total number of pebbles on the vertices $v_5, v_7,$ and $v_9$ contain at most nine. Thus, one of the vertices of $\{v_6, v_8\}$ contains at least four pebbles. If one of the vertices of $\{v_5, v_9\}$ contains two or three pebbles then we can move a pebble to $v_2$ easily. So, assume both $v_5$ and $v_9$ contain at most one pebble each. If $v_{11}$ contains a pebble and if $v_7$ contains at least six pebbles then we can move a pebble to $v_2$. So assume that $v_7$ contains at most five pebbles. In this case also, we can move a pebble to $v_2$, since the total number of pebbles on the vertices $v_6$ and $v_8$ is at least nine. Thus, we assume that $v_7$ contains at most three pebbles. In a similar way, we may assume that both $v_5$ and $v_9$ contain at most three pebbles each.

Suppose all the three vertices $v_5, v_7,$ and $v_9$ contain two or three pebbles each. Then, the vertices $v_6$ and $v_8$ totally contain at least six pebbles. So, we can move at least one pebble to $v_{11}$ using the pebbles at $v_6$ and $v_8$. So, we can move a pebble to $v_2$. Thus, assume that two of the vertices of $\{v_5, v_7, v_9\}$, say $v_5$ and $v_9$ contain at least two or three pebbles each. Then, the total number of pebbles on the vertices $v_6$ and $v_8$ is at least eight. If $v_7$ contains one pebble then we can move a pebble to $v_2$—a contradiction to case(iii). Now, exactly one vertex from $\{v_5, v_7, v_9\}$, say $v_5$ contains two or three pebbles. Then the total number of pebbles on the vertices $v_6$ and $v_8$ is at least ten. If either $v_{11}$ or $v_4$ contains a pebble then we can move a pebble to $v_2$ easily. So assume both $v_4$ and $v_{11}$ contain zero pebbles on it. Then, the total number of pebbles on the
Suppose all the three vertices \(v_5, v_7, v_9\) contain two or three pebbles each. Then, the vertices \(v_6\) and \(v_8\) totally contain at least six pebbles. So, we can move at least one pebble to \(v_{11}\) using the pebbles at \(v_6\) and \(v_8\). So, we can move a pebble to \(v_2\). Thus, assume that two of the vertices \(v_5, v_7, v_9\) contain at least two or three pebbles each. Then, the total number of pebbles on the vertices \(v_6\) and \(v_8\) is at least eight. If \(v_7\) contains one pebble then we can move a pebble to \(v_2\), a contradiction to case(iii). Now, exactly one vertex from \(v_5, v_7, v_9\), say \(v_5\) contains two or three pebbles. Then the total number of pebbles on the vertices \(v_6\) and \(v_8\) is at least ten. If either \(v_{11}\) or \(v_4\) contains a pebble then we can move a pebble to \(v_2\) easily. So assume both \(v_4\) and \(v_{11}\) contain zero pebbles on it. Then, the total number of pebbles on the vertices \(v_6\) and \(v_8\) is at least twelve. If \(v_7\) contains one pebble then also we can move a pebble to \(v_2\). Otherwise, we can move a pebble to \(v_2\) easily, since the total number of pebbles on the vertices \(v_6\) and \(v_8\) is at least thirteen. Thus assume that all the three vertices \(v_5, v_7, v_9\) contain at most one pebble each. Now, the total number of pebbles on the vertices \(v_6\) and \(v_8\) is at least twelve. If both \(v_7\) and \(v_{11}\) contain one pebble each then we can move a pebble to \(v_2\). So, assume \(v_{11}\) has zero pebbles on it. Suppose all the three vertices \(v_5, v_7, v_9\) contain one pebble each. One of the vertices \(\{v_6, v_8\}\) receives at least six pebbles and hence we can move a pebble to \(v_2\). So, we may assume \(v_7\) has zero pebbles on it. If both \(v_4\) and \(v_{10}\) have one pebble each then we can move a pebble to \(v_2\), since the total number of pebbles on the vertices \(v_6\) and \(v_8\) is fourteen. For the other distributions, it is easy to see that, a pebble can be moved to \(v_2\), a contradiction to case(iii). This implies that \(f(J_{2,5}) \leq 18\).

Therefore, \(f(J_{2,5}) = 18\).

**Theorem 2.6** For the Jahangir graph \(J_{2,6}\), \(f(J_{2,6}) = 21\).

**Proof** Put fifteen pebbles at \(v_6\), three pebbles at \(v_{10}\) and one pebble each at \(v_8\) and \(v_{12}\). Then, we cannot move a pebble \(v_2\). Thus, \(f(J_{2,6}) \geq 21\).

Now, consider a distribution of twenty one pebbles on the vertices of \(J_{2,6}\). Consider the sets \(S_1 = \{v_1, v_3, v_5, v_7, v_9, v_{11}\}\) and \(S_2 = \{v_2, v_4, v_6, v_8, v_{10}, v_{12}\}\).

**Case (i)** Suppose we cannot move a pebble to the vertex \(v_{13}\).

The vertices of \(S_1\) contain at most one pebble each and the vertices of \(S_2\) contain at most three pebbles each. Also, note that, no two consecutive vertices of \(S_2\) contain two or three pebbles each and if a vertex of \(S_2\) contains two or three pebbles then the neighbors of that vertex contain zero pebbles. Also, if a vertex of \(S_1\) contains one
pebble then the neighbors of that vertex contain at most one pebble each. Thus, in any
distribution, $J_{2,6}$ contains at most twelve pebbles so that a pebble could not be moved
to $v_{13}$—a contradiction to the total number of pebbles placed over the vertices of $J_{2,6}$.
So, our assumption (case(i)) is wrong.

**Case (ii)** Suppose we cannot move a pebble to a vertex of $S_1$. Without loss of
generality, let $v_1$ be the target vertex.

Clearly, the neighbors $v_2$, $v_{12}$, and $v_{13}$ of $v_1$ contain at most one pebble each. The
vertices of $S_1 - \{v_1\}$ contain at most three pebbles each and the vertices of $S_2 - \{v_2, v_{12}\}$ contain at most seven pebbles each. Also, note that, no two vertices of $S_1 - \{v_1\}$ contain two or three pebbles each and if a vertex of $S_1 - \{v_1\}$ contains two or three
pebbles then the neighbors of that vertex contain at most three pebbles each. Also, no
two vertices of $S_2 - \{v_2, v_{12}\}$ contain four or more pebbles each and if a vertex of $S_2 - \{v_2, v_{12}\}$ contains four or more pebbles then the neighbors of that vertex contain at most one pebble each.

Suppose $v_{13}$ has a pebble on it. Then clearly the vertices of $S_1 - \{v_1\}$ contain at most one pebble each and the vertices of $S_2 - \{v_2, v_{12}\}$ contain at most three pebbles each. Thus in any
distribution, $J_{2,6}$ contains at most eleven pebbles so that a pebble could not be moved to $v_1$—a contradiction to the total number of pebbles placed over the vertices of $J_{2,6}$. So, assume that $v_{13}$ has zero pebbles on it.

Suppose $v_2$ has a pebble on it. Clearly, the neighbor $v_3$ contains at most one pebble and
the vertex $v_4$ contains at most three pebbles. If $v_5$ has two or three pebbles, then the
vertices of $S_1 - \{v_1, v_5\}$ contain at most one pebble each. Thus, one of the vertices of
$\{v_8, v_{10}\}$ contains at least four pebbles and hence, we can move a pebble to $v_1$ through
$v_{13}$—a contradiction to case(ii). So, assume that $v_5$ has at most one pebble. In a similar
way, we may assume that the vertices $v_7$, $v_9$, and $v_{11}$ contain at most one pebble each.
Thus, one of the vertices of $\{v_6, v_8, v_{10}\}$ contains at least six pebbles, say $v_8$. If one of
the neighbors of $v_8$ contains one pebble then we can move a pebble to $v_1$ through $v_{13}$-a contradiction to case(ii). Otherwise, $v_8$ contains at least eight pebbles. So, we can move a pebble to $v_1$. So, assume that $v_2$ has zero pebbles on it. In a similar way, we may assume that the vertex $v_{12}$ has zero pebbles.

Suppose $v_3$ has two or three pebbles on it. Clearly, the vertices of $S_1$–{$v_1, v_3$} contain at most one pebble each and the vertices of $S_2$–{$v_2, v_{12}$} contain at most three pebbles each. Also, if a vertex of $S_2$–{$v_2, v_{12}$} contains two or three pebbles then the proceeding and succeeding vertices in $S_2$ contain at most one pebble. Thus in any distribution, $J_{2, 6}$ contains at most eleven pebbles so that a pebble could not be moved to $v_1$-a contradiction. So, assume $v_3$ has at most one pebble. In a similar way, we may assume that, the vertices $v_5, v_7, v_9$, and $v_{11}$ contain at most one pebble each. Clearly, the total number of pebbles on the vertices $v_4$, $v_6$, $v_8$, and $v_{10}$ is at least sixteen. Thus, two of the vertices of {$v_4, v_6, v_8, v_{10}$} contain four or more pebbles each or one of the vertices {$v_4, v_6, v_8, v_{10}$} receives at least eight pebbles. Thus, we can always move a pebble to $v_1$-a contradiction to case(ii). So, our assumption (case(ii)) is wrong.

Case (iii) Suppose we cannot move a pebble to a vertex of $S_2$. Without loss of generality, let $v_2$ be the target vertex.

Clearly, the neighbors $v_1$, and $v_3$ of $v_2$ contain at most one pebble each and the vertices $v_4, v_{12}$, and $v_{13}$ contain at most three pebbles each. The vertices of $S_1$ – {$v_1, v_3$} contain at most seven pebbles each and the vertices of $S_2$ – {$v_2, v_4, v_{12}$} contain at most fifteen pebbles each. Also, note that, no two vertices of $S_1$ – {$v_1, v_3$} contain four or more pebbles each and if a vertex of $S_1$ – {$v_1, v_3$} contain four or more pebbles then the neighbors of that vertex contain at most three pebbles each. Also, no two vertices of $S_2$ – {$v_2, v_4, v_{12}$} contain eight or more pebbles each and if a vertex of $S_2$ – {$v_2, v_4, v_{12}$} contains eight or more pebbles then the neighbors of that vertex contain at most three pebbles each.
Suppose $v_3$ has a pebble on it. Clearly, the vertices $v_4$ and $v_{13}$ contain at most one pebble each and at most one of the vertices of $S_1 - \{v_1, v_3\}$ contains two or three pebbles. Thus, the total number of pebbles on the vertices $v_6, v_8, v_{10},$ and $v_{12}$ is at least twelve. So, in any distribution of these twelve pebbles on the vertices $v_6, v_8, v_{10},$ and $v_{12},$ we must have the following cases: either one of the vertices receives eight or more pebbles or two of the vertices of $\{v_6, v_8, v_{10}, v_{12}\}$ receive four or more pebbles each or all the vertices receive two or three pebbles each. So, we can move a pebble to $v_2.$ So, assume that $v_3$ has zero pebbles on it. In a similar way we may assume that $v_1$ has zero pebbles on it.

Suppose $v_{13}$ has two or three pebbles on it. Clearly the vertices of $S_1 - \{v_1, v_3\}$ contain at most three pebbles each and no two vertices of $S_1 - \{v_1, v_3\}$ contain two or three pebbles each. Thus, we can move at least two pebbles to $v_{13},$ since the total number of pebbles on the vertices of $\{v_4, v_6, v_8, v_{10}, v_{12}\}$ is at least thirteen (note that the vertices $v_4$ and $v_{12}$ contain at most one pebble each). So, assume that $v_{13}$ contains at most one pebble.

Suppose $v_4$ has two or three pebbles on it. Then, the vertex $v_5$ contains at most three pebbles and the vertex $v_{13}$ contains at most one pebble. Thus the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}, v_{12}\}$ is twelve. So, we can move a pebble to $v_3$ and then move a pebble to $v_2$ using the pebbles at $v_4$ - a contradiction to case(iii). So, assume that $v_4$ has at most one pebble on it. In a similar way, we may assume that $v_{12}$ has at most one pebble.

Suppose $v_7$ has four or more pebbles (at most seven pebbles) on it. Then the total number of pebbles on the vertices $\{v_5, v_7, v_9, v_{11}\}$ is at most ten. Thus the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least nine. If $v_{13}$ has one pebble on it then we can move a pebble to $v_2$ through $v_{13}$ using the pebbles at $v_7.$ So, assume that $v_{13}$ has zero pebbles on it. Also, if $v_7$ contains six or seven pebbles then we can
move a pebble to $v_2$. So, assume that the vertex $v_7$ contains at most five pebbles. Thus the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least nine. If one of the vertices of $\{v_6, v_8, v_{10}\}$ contains two or three pebbles then we can move a pebble to $v_2$. Otherwise, it is easy to see that, we can move a pebble to $v_2$. So, assume $v_7$ contains at most three pebbles. In a similar way, we assume that the vertices $v_5$, $v_9$, and $v_{11}$ contain at most three pebbles each.

If the four vertices $v_5$, $v_7$, $v_9$ and $v_{11}$ contain two or three pebbles each then clearly we can move a pebble to $v_2$. So, assume that at most three of them contain two or three pebbles each. Suppose the vertices $v_5$, $v_7$, and $v_9$ contain two or three pebbles each then the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least nine. Thus we can move a pebble to $v_2$ easily. Next assume that any two of the vertices of $\{v_5, v_7, v_9, v_{11}\}$ contain two or three pebbles each. Then the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least ten. If $v_{13}$ contains one pebble then clearly we can move a pebble to $v_2$. Otherwise, we can move a pebble to $v_2$ easily according to the distribution of eleven pebbles at $v_6$, $v_8$, and $v_{10}$.

Assume one of the vertices from the set $\{v_5, v_7, v_9, v_{11}\}$ contains two or three pebbles on it. Then the total number of pebbles on the vertices of $\{v_6, v_8 ,v_{10}\}$ is at least thirteen. So, we can move a pebble to $v_2$ if $v_{13}$ contains one pebble on it. If not, then the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least fourteen. Thus, one of the vertices of $\{v_6, v_8 , v_{10}\}$ contains at least four pebbles, say $v_8$.

Suppose $v_8$ has six or seven pebbles. If one of the neighbors of $v_8$ contains one pebble, then we can move a pebble to $v_2$. Otherwise, the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least sixteen. So, we can move a pebble to $v_2$. Thus assume $v_8$ contains at most five pebbles. Then clearly we can move a pebble to $v_2$, since the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least fourteen. So, assume that $v_8$ contains at most three pebbles. In a similar way, we may assume
that the vertices \( v_6 \) and \( v_{10} \) contain at most three pebbles each. But, we get a contradiction to the total number of pebbles placed on \( J_{2,6} \). So, assume that the vertices \( v_5, v_7, v_9 \) and \( v_{11} \) contain at most one pebble each. Thus, the total number of pebbles on the vertices of \( \{v_6, v_8, v_{10}\} \) is at least fourteen. A similar argument shows that \( v_8 \) contain at least four pebbles and so we can move a pebble to \( v_2 \). So, our assumption is wrong.

Thus, \( f(J_{2,6}) \leq 21 \).

Therefore, \( f(J_{2,6}) = 21 \).

**Theorem 2.7** For the Jahangir graph \( J_{2,7} \), \( f(J_{2,7}) = 23 \).

**Proof** Put fifteen pebbles at \( v_6 \), three pebbles at \( v_{10} \) and one pebble each at \( v_8, v_{14}, v_{12}, \) and \( v_{13} \). Then, we cannot move a pebble to \( v_2 \). Thus, \( f(J_{2,7}) \geq 23 \).

As we argued in previous theorems, we can show that \( f(J_{2,7}) \leq 23 \).

Hence \( f(J_{2,7}) = 23 \).

**Conjecture 2.8** For the Jahangir graph \( J_{2,m} \) (\( m \geq 8 \)), \( f(J_{2,m}) = 2m + 10 \).

**Conclusion.** In this paper, we have computed the pebbling number for Jahangir graph \( J_{2,m} \) where \( 3 \leq m \leq 7 \). We strongly believe that the pebbling number of the Jahangir graph \( J_{2,m} \) where \( m \geq 7 \) is \( f(J_{2,m}) = 2m + 10 \).

**References:**


