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ON SAME ORDER TYPE GROUPS

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ABSTRACT

A group is of fundamental importance in the algebraic structure. Atagün introduced the concept of a group matrix in the 9(2), 2005 of this journal. We call a class of groups of equivalent group matrices the same order type groups. Atagün asserted that two groups with equivalent group matrices are isomorphic. In this paper we give some counterexamples to this assertion. Then we continue to discuss the same order type groups and a well-known Thompson's problem related.

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1. INTRODUCTION

Throughout this paper the capital letters $G, H, \cdots$ will always denote finite groups, $o(x)$ the order of a group element x, and $|X|$ the cardinality of a set X. Groups which are, from the point of view of algebraic structure, essentially the same are said to be isomorphic. The modern group theory is created by E. Cayley , he posed a very hard problem that what are groups of order n, that is classifying groups of order n. We call a subgroup N of G is normal if every conjugate subgroup of N is same to N. A group G called simple group if G has only two normal subgroups 1 and G. Simple groups are likely to primes of integers, which are bracket of the building of groups. In Feb. 1981, the classification of the finite simple groups was called completed, representing one of the most remarkable achievements in the history of mathematics, and proved it in 2003 (the efforts of several hundred mathematicians over a period of 50 years, full proof covered something between 5000 and 10000 journal pages).

Atagün introduced the concept of a group matrix in the 9(2),2005 of this journal [1].

Definition 1.1 Let G be a groups of order n. We will denote the elements which order k in G by g_k and g_{k_j} will denote the j-th element which order k in G. If the matrix $\theta = \{a_{ij}\}_{n\times n}$ consists of the elements $a_{11}=1, a_{12}=a_{13}=\cdots=a_{1n}=a_{21}=a_{31}=\cdots=a_{n1}=0$ and $g_{k_j}=a_{k(j+1)}$ where $1\le j\le n-1, 2\le k\le n$, then we will call it a group matrix of G and denote this by θ_G.

Let G and H be groups of the same finite order n and these group matrices are θ_G and θ_H, respectively. If for every $1\le i\le n$ the i-th rows of θ_G and θ_H have non-zero elements of same number, then we will call these matrices are equivalent and denote $\theta_G \approx \theta_H$. If G and H have equivalent matrices, then these have same numbers of elements of each order. In fact we can express the equivalent group matrix into the same order type groups.

Definition 1.2 For each finite group G and each integer $d \geq 1$, let $G(d) = \{x \in G \mid x^d = 1\}$. Groups G and H are of the same order type if and only if $|G(d)| = |H(d)|, d = 1, 2, \cdots$.

Atagün asserted that two groups with equivalent group matrices are isomorphic, that is two groups with same order type are isomorphic. In the sequel we give some counterexamples to this assertion.

2. COUNTER EXAMPLES

Obviously, two isomorphic groups are of same order type necessarily. Conversely, the clam is not always true. In this part, we give some counterexamples to confirm the following fact.

Theorem 2.1 If $\theta_G \approx \theta_H$, then G is not necessary to isomorphic to H.

Next we give some examples.

Example 2.2 Let $G = \langle a, b, c \mid a^4 = 1, b^2 = c^2 = 1, b^a = b, c^a = c, c^b = ca^2 \rangle$ and $H = \langle a, b, c \mid a^4 = 1, b^2 = c^2 = 1, b^a = b, c^a = ca^2, c^b = c \rangle$.
Then G and H are both of order 16. The number of order 2 in G and H is 7, and one of order 4 is 8. They have the same order type. Since the center $Z(G)$ of G is isomorphic to Z_4, and $Z(H)$ isomorphic to $Z_2 \times Z_2$, it follows that G is not isomorphic to H.

Example 2.3 Let the prime $p > 2$. Suppose that G is an elementary abelian p-group whose order is more than p^2. Let H be a group with $exp(G) = p$ and $|G| = |H|$. Then G and H have the same order type. In general there exists a group H such that G is not isomorphic to H. For example, suppose that $|G| = 27$ and $H = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, a^b = a, a^c = ab^{-1}, c^b = c \rangle$.
Then $Z(H) \cong Z_3$. So G is not isomorphic to H.

There exist lots of groups with same order type in nilpotent groups. It seems that the examples with the same order type in non-solvable groups are very rare. In fact, we cannot find an example of non-abelian simple group S such that S and other groups have a same order type. This follows from Shi's conjecture that if a finite group G has same order and the set of orders of elements of a simple group S, then $G \cong S$. Shi's conjecture posed in the 1980's, and collected in the book of unsolved problem in group theory(see the problem 12.39, [2]). Now Shi's conjecture is confirmed by some authors including Shi, Mazurov, Vasilev and Grechkoseeva, etc [4]. The next example of non-solvable groups is due to J.G. Thompson.

Example 2.4 Let M_{23} be the Mathiu sporadic simple group. Suppose that $G = Z_{2^4} : A_7$ and $H = L_3(4) : Z_2$ are both the maximal subgroups of the M_{23}. Then G and H have the same order type (refer to [3]).

3. THOMPSON'S PROBLEM

In this part, we introduce the well-known Thompson's problem. Also we give a positive answer for finite nilpotent and super-solvable groups to Thomposn's problem. Given a finite group G, how to judge the solvability of G? The famous Odd Order Theorem proved that all finite groups of odd order are solvable. In 1987 Professor W.J. Shi reported his conjecture (that is above conjecture on finite simple groups) to Prof. J.G. Thompson. In their communications J. G. Thompson posed his problem:

Thompson Problem (Problem 12.37, [2]) Suppose G and H are two groups of the same order type. Suppose that G is solvable. Is it true that H is also necessarily solvable?

That is, for the groups G of even order, we can not judge the solvability of G only using the order of G, but we may judge it using the order type of G if the answer of above Problem is in the affirmative. In Thompson's private letter he pointed out that `` I have talked with several mathematicians concerning groups of the same order type. The problem arose initially in the study of algebraic number fields, and is of considerable interest".

Next we give a positive answer for finite nilpotent and super-solvable groups to Thomposn's problem.

4. CONCLUSIONS

Proposition 1 Suppose G and H are two groups of the same order type. Suppose that G is nilpotent. Then H is also nilpotent.

Proof. Since G and H have the same order type, it follows that $|G| = |H|$ and the number of elements of each same order of G is same as one of H. Let

$|G|=p_1^{s_1}\cdots p_k^{s_k}$. Suppose that P_i is the unique Sylow p_i-subgroup of G for $1\le i\le k$. Then the number of p_i-elements of G is $p_i^{s_i}$. So the number of p_i-elements of H is also $p_i^{s_i}$. It must lead that H has a unique Sylow p_i-subgroup, and then H is nilpotent.

Recall that a group G is called super-solvable if every chief factor of G is of prime order.

Proposition 2 Suppose G and H are two groups of the same order type. Suppose that G is super-solvable. Then H is solvable.

Proof. Since G is super-solvable, there exists a normal series

$$1<H_1<H_2<\cdots<H_s=G$$

of G such that every H_i is a Hall subgroup of G and H_i/H_{i-1} of order of a power of a prime. Let $|G|=p_1^{k_1}p_2^{k_2}\cdots p_s^{k_s}$. We can assume that $|H_i|=p_1^{k_1}p_2^{k_2}\cdots p_i^{k_i}$ for $i=1,2,\cdots s$. It is clear that $|G(p_1^{k_1}p_2^{k_2}\cdots p_i^{k_i})|=p_1^{k_1}p_2^{k_2}\cdots p_i^{k_i}$, so $|H(p_1^{k_1}p_2^{k_2}\cdots p_i^{k_i})|=p_1^{k_1}p_2^{k_2}\cdots p_i^{k_i}$. On the other hand, if $|G(d)|=d$, then G has a normal subgroup of order d (This assertion is a conjecture of Frobenius, it proved by N. Liyori and H. Yamaki in 1991 [5] using the classification of the finite simple groups). Thus H also has normal Hall subgroups whose orders are $p_1^{k_1}p_2^{k_2}\cdots p_i^{k_i}$ for $i=1,2,\cdots s$. It follows that H is solvable.

By far a counter-example to Thompsom problem is not found. Recently R. Shen and W. Shi proved that if G has a non-connected prime graph, then this problem is true (see [6]).

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